# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 1**

**Question:**

A uniform circular disc of mass 2 kg and radius 0.7 m is rotating in a horizontal plane about a smooth fixed vertical axis through its centre. Calculate its kinetic energy when it is rotating at $5 \text{ rad s}^{-1}$.

**Solution:**

$$\text{K.E.} = \frac{1}{2}I\omega^2$$

$$= \frac{1}{2} \times \left(\frac{1}{2} \times 2 \times 0.7^2\right) \times 5^2$$

$$= 6.125 \text{ J}$$

The M.I. of a circular disc is in the formula book.

The kinetic energy is 6.125 J.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 2**

**Question:**

A uniform circular disc of mass 4 kg and radius 0.25 m has particles of mass 0.1 kg, 0.2 kg and 0.8 kg attached to it at points which are 0.2 m, 0.1 m and 0.15 m respectively from the centre of the disc. The loaded disc is rotating at 4 rad $s^{-1}$ about a fixed smooth vertical axis through its centre perpendicular to the disc.

**a** Calculate the kinetic energy of the loaded disc.

The disc is now brought to rest.

**b** Write down the work done by the retarding force.

**Solution:**

**a** M.I. of disc and particles $= \frac{1}{2} \times 4 \times 0.25^2 + 0.1 \times 0.2^2 + 0.2 \times 0.1^2 + 0.8 \times 0.15^2$

$= 0.149 \text{ kg m}^2$

$\text{K.E.} = \frac{1}{2} I\omega^2$

$= \frac{1}{2} \times 0.149 \times 4^2$

$= 1.192 \text{ J}$

The kinetic energy is 1.19 J (3 s.f.)

**b** The work done by the retarding force is 1.19 J (3 s.f.) ← Work done by the retarding force = loss of K.E.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

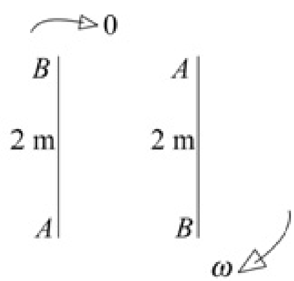
**Exercise A, Question 3**

**Question:**

A uniform rod $AB$ of mass 2.5 kg and length 2 m can rotate in a vertical plane about a fixed smooth horizontal axis through $A$ perpendicular to $AB$. Initially it is at rest with $B$ vertically above $A$. It is then slightly disturbed and begins to rotate.

**a** Calculate the potential energy lost by the rod when it is horizontal.
**b** Write down the kinetic energy of the rod when it is horizontal.
**c** Calculate the angular speed of the rod when $B$ is vertically below $A$.

**Solution:**




**a** P.E. lost $= mgh = 2.5 \times 9.8 \times 1$

$= 24.5$

The potential energy lost is 24.5 J

**b** The kinetic energy of the rod when it is horizontal is 24.5 J

**c** M.I. of the rod about the axis through $A$

$= \frac{4}{3} \times 2.5 \times 1^2$ ← The formula for the required M.I. can be obtained from the formula book.

$= \frac{10}{3}$

$\frac{1}{2} I\omega^2 = 2.5g \times 2$ ← K.E. gained = P.E. lost

$\frac{1}{2} \times \frac{10}{3}\omega^2 = 2.5g \times 2$ ← You can work from the start or from the horizontal position. The former is easier.

$\omega^2 = \frac{5 \times 9.8 \times 6}{10}$

$\omega = 5.422\ldots$

The angular speed is $5.42 \text{ rad s}^{-1}$ (3 s.f.)

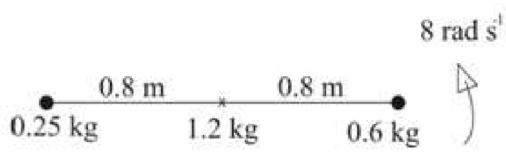
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 4**

**Question:**

A uniform rod of length 1.6 m and mass 1.2 kg has particles of mass 0.25 kg and 0.6 kg attached, one at each end. The rod is rotating about a fixed smooth vertical axis perpendicular to the rod with angular speed $8 \text{ rad s}^{-1}$. Calculate the kinetic energy of the rod when the axis passes through the mid-point of the rod.

**Solution:**




M.I. of rod and particles about the given axis through the mid-point

$$= \frac{1}{3} \times 1.2 \times 0.8^2 + 0.25 \times 0.8^2 + 0.6 \times 0.8^2$$

$$= 0.8 \text{ kg m}^2$$

$$\text{K.E.} = \frac{1}{2} I\omega^2 = \frac{1}{2} \times 0.8 \times 8^2$$

$$= 25.6$$

The kinetic energy is 25.6 J

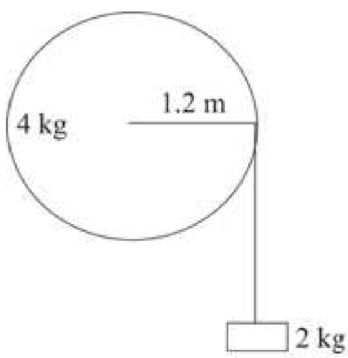
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 5**

**Question:**

A pulley wheel of mass 4 kg and radius 1.2 m is free to rotate in a vertical plane about a fixed smooth horizontal axis through the centre of the pulley and perpendicular to the pulley. A block of mass 2 kg hangs freely attached to one end of a rope. The other end of the rope is attached to a point on the rim of the pulley and the rope is wound several times around the pulley. Initially the block is hanging 5 m above horizontal ground. The block is then released from rest. The pulley wheel can be modelled as a uniform disc, the block as a particle and the rope as a light inextensible string. Calculate the angular speed of the pulley at the instant when the block hits the ground.

**Solution:**




P.E. lost by block $= 2g \times 5$

$= 98\text{ J}$

The block falls 5 m.

K.E. gained by block and pulley $= \frac{1}{2}I\omega^2 + \frac{1}{2}m(r\omega)^2$

When the angular speed of the pulley is $\omega$, the block's (linear speed) is $r\omega$.

$$\therefore \frac{1}{2}\times\left(\frac{1}{2}\times 4\times 1.2^2\right)\omega^2 + \frac{1}{2}\times 2\times 1.2^2\omega^2 = 98$$

$$1.44\omega^2 + 1.44\omega^2 = 98$$

$$\omega^2 = \frac{98}{2.88}$$

$$\omega = 5.833\ldots$$

The angular speed is $5.83\text{ rad s}^{-1}$ (3 s.f.)

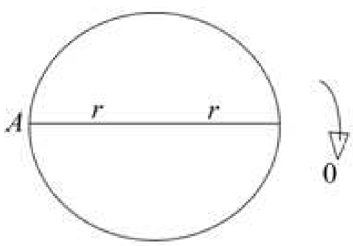
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

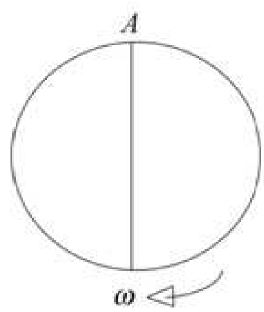
**Exercise A, Question 6**

**Question:**

A uniform disc of radius $r$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to the disc through a point $A$ of its edge. The disc is released from rest with the diameter through $A$ horizontal. Find the angular speed of the disc when this diameter is vertical.

**Solution:**




$\frac{1}{2}I\omega^2 = mgh$ ← K.E. gained = P.E. lost

M.I. of disc about axis through $A$

$= \frac{1}{2}mr^2 + mr^2$ ← Use the parallel axes theorem.

$\frac{1}{2} \times \frac{3}{2}mr^2\omega^2 = mgr$ ← Use $m$ for the mass of the disc. It will cancel.

$\omega^2 = \frac{4g}{3r}$

$\omega = 2\sqrt{\frac{g}{3r}}$

The angular speed is $2\sqrt{\frac{g}{3r}}$ ← Any equivalent answer is acceptable.

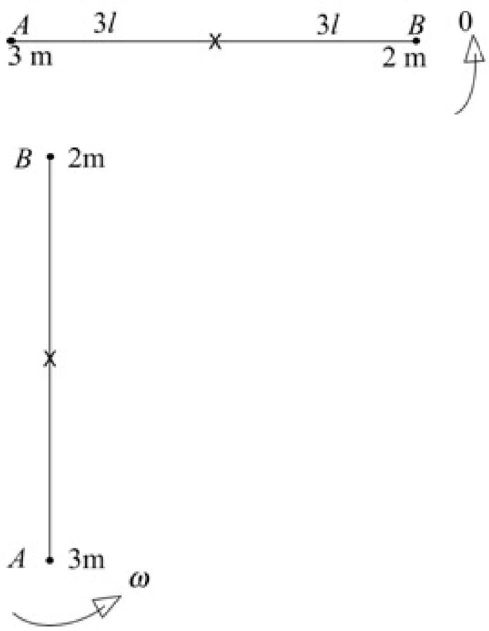
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 7**

**Question:**

A uniform rod $AB$ of mass $m$ and length $6l$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to $AB$ through its mid-point. Particles of masses $3m$ and $2m$ are attached to ends $A$ and $B$ respectively. The rod is held at rest with $AB$ horizontal and then released. Find, in terms of $l$ and g, the angular speed of the rod when $AB$ is vertical.

**Solution:**




M.I. of rod and particles about given axis

$$= \frac{1}{3}m(3l)^2 + 3m(3l)^2 + 2m(3l)^2$$

$$= 48\,ml^2$$

$$\frac{1}{2}I\omega^2 = mgh$$

K.E. gained = P.E. lost

$$\frac{1}{2} \times 48ml^2\omega^2 = 3mg \times 3l - 2mg \times 3l$$

$$24ml^2\omega^2 = 3mgl$$

$$\omega^2 = \frac{g}{8l}$$

$$\omega = \sqrt{\frac{g}{8l}} = \frac{1}{2}\sqrt{\frac{g}{2l}}$$

Any equivalent answer is acceptable.

# Solutionbank M5

## Edexcel AS and A Level Modular Mathematics

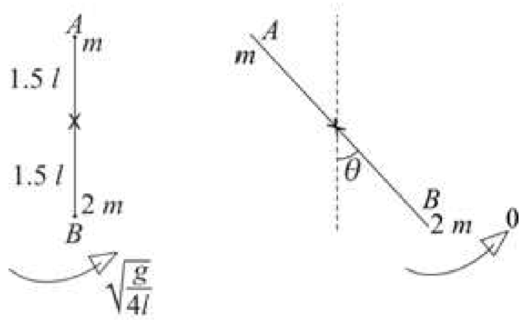
**Exercise A, Question 8**

**Question:**

A uniform rod $AB$ of mass $m$ and length $3l$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to $AB$ through its mid-point. Particles of masses $m$ and $2m$ are attached to ends $A$ and $B$ respectively. The rod is initially vertical with $B$ below $A$.

It then receives an impulse and starts to rotate with angular speed $\sqrt{\frac{g}{4l}}$. Calculate, to the nearest degree, the angle between $AB$ and the downward vertical when the rod first comes to rest.

**Solution:**




M.I. of loaded rod about given axis

$$= \frac{1}{3}m\times(1.5l)^2 + m\times(1.5l)^2 + 2m\times(1.5l)^2$$

$$= 7.5ml^2$$

Rod comes to rest when the angle between $AB$ and the downward vertical is $\theta$:

$$2mg\times 1.5l(1-\cos\theta) - mg\times 1.5l(1-\cos\theta) = \frac{1}{2}\times 7.5ml^2\left(\sqrt{\frac{g}{4l}}\right)^2$$ ← P.E. gained = K.E. lost

$$1.5\,mlg(1-\cos\theta) = 3.75ml^2\times\frac{g}{4l}$$

$$1.5(1-\cos\theta) = \frac{3.75}{4}$$

$$1-\cos\theta = \frac{3.75}{6}$$

$$\cos\theta = 1-\frac{3.75}{6}$$

$$\theta = 67.97\ldots$$

The angle is $68^\circ$ (nearest degree)

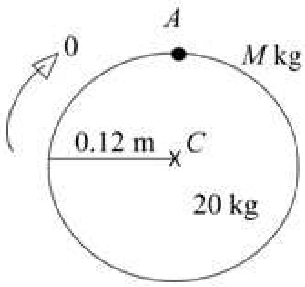
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

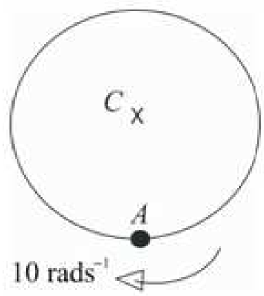
**Exercise A, Question 9**

**Question:**

A uniform circular disc of mass 20 kg and radius 12 cm is free to rotate about a fixed smooth horizontal axis through its centre $C$ perpendicular to the disc. A particle of mass $M$ kg is attached to point $A$ of the rim of the disc. Initially the disc is at rest with $A$ vertically above $C$. The disc is then slightly disturbed. The greatest angular speed of the disc in the subsequent motion is $10 \text{ rad s}^{-1}$. Find the value of $M$.

**Solution:**




Watch the units!

The greatest angular speed will occur when $A$ is vertically below $C$ as the loss of P.E. is greatest here.

M.I. of disc and particle about the given axis

$$= \frac{1}{2} \times 20 \times 0.12^2 + M \times 0.12^2$$

$$= 0.144 + 0.0144M$$

$$\frac{1}{2} I\omega^2 = Mg \times 0.24$$

Gain of KE = loss of PE

$$\frac{1}{2}(0.144 + 0.0144M) \times 10^2 = Mg \times 0.24$$

$$14.4 + 1.44M = M \times 9.8 \times 0.48$$

$$M = \frac{14.4}{(9.8 \times 0.48 - 1.44)}$$

$$M = 4.411\ldots$$

$$\therefore M = 4.41 \text{ (3 s.f.)}$$

# Solutionbank M5

## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 10**

**Question:**

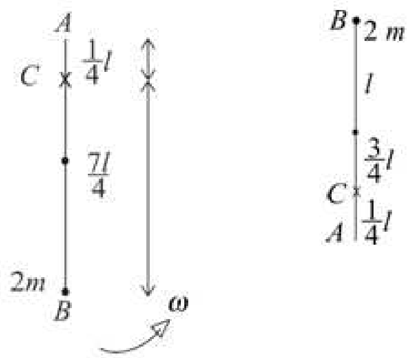
A uniform rod $AB$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to $AB$ through point $C$ of the rod, where $AC = \frac{1}{4}l$. The rod has mass $m$ and length $2l$, and a particle of mass $2m$ is attached to end $B$. Initially the rod is hanging in equilibrium with $B$ vertically below $A$. The rod then receives an impulse and starts to rotate with angular speed $\omega$. In the subsequent motion, the rod moves in a complete circle. The least possible value of $\omega$ is $\Omega$.

**a** Show that $\Omega = 4\sqrt{\frac{51g}{337l}}$.

The initial angular speed is $2\Omega$.

**b** Find the speed of the particle as it passes vertically above $C$.

**Solution:**




**a** M.I. of rod and particle about given axis through $C$

$$= \frac{1}{3}ml^2 + m\times\left(\frac{3}{4}l\right)^2 + 2m\times\left(\frac{7l}{4}\right)^2$$

Use the parallel axes theorem.

$$= \frac{337}{48}ml^2$$

For least $\omega$, angular speed $= 0$ when $B$ is vertically above $A$.

$$\text{At top: P.E. gained} = mg\times 2\times\frac{3}{4}l + 2mg\times 2\times\frac{7}{4}l$$

$$= \frac{17}{2}mgl$$

$$\therefore \frac{1}{2}\times\frac{337}{48}ml^2\Omega^2 = \frac{17}{2}mgl$$

K.E. lost = P.E. gained

$$\Omega^2 = \frac{48}{337}\times 17\frac{g}{l}$$

$$\Omega = 4\sqrt{\frac{51g}{337l}}$$

**b**

$$\frac{17mgl}{2} = \frac{1}{2}\times\left(\frac{337}{48}ml^2\right)\times(2\Omega)^2 - \frac{1}{2}\times\left(\frac{337}{48}ml^2\right)\omega^2$$

The energy equation now includes the K.E. at the top.

$$\frac{337}{48}ml^2\omega^2 = \frac{337}{48}ml^2\times 4\Omega^2 - 17mgl$$

$$\frac{337}{48}ml^2\omega^2 = 4\times 17mgl - 17mgl$$

From **a** $17mgl = \frac{337ml^2}{48}\Omega^2$

$$\frac{337}{48}l\omega^2 = 51g$$

$$\omega^2 = \frac{48\times 51g}{337l}$$

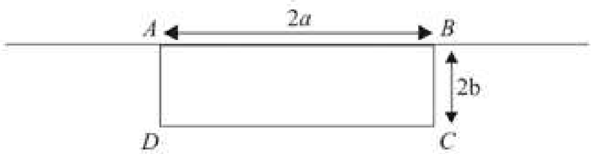
$$\omega = 12\sqrt{\frac{17g}{337l}}$$

Any equivalent form is acceptable.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 11**

**Question:**




The diagram shows a sign which hangs outside a shop. The sign is a thin rectangular metal plate which is free to rotate about a fixed smooth horizontal axis which lies along the side $AB$. The lengths of $AB$ and $BC$ are $2a$ and $2b$ respectively. The sign can be modelled as a uniform rectangular lamina. The sign is hanging freely below the axis when it receives a blow and starts to rotate with angular speed $k\sqrt{\frac{g}{b}}$.

**a** Find the least value of $k$ for which the sign makes complete revolutions.

**b** If $k = 1.5$, find the angle $BC$ makes with the upward vertical when the sign first comes to rest.

**Solution:**

**a** M.I. of sign about axis along $AB = \frac{4}{3}mb^2$

For the least initial angular speed for complete revolutions:

$$\frac{1}{2}\left(\frac{4}{3}mb^2\right)\left(k\sqrt{\frac{g}{b}}\right)^2 = mg \times 2b$$

$$\frac{2}{3}mb^2 \times k^2 \frac{g}{b} = 2mgb$$

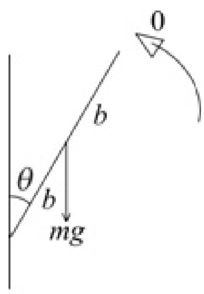
$$k^2 = 3$$

The least value of $k$ is $\sqrt{3}$

From the formula book, letting the mass of the rectangle be $m$.

The angular speed at the top will be zero.

**b**




$$\text{Initial K.E.} = \frac{1}{2} \times \frac{4}{3}mb^2 \times \left(1.5\sqrt{\frac{g}{b}}\right)^2$$

$$= 1.5mgb$$

$$\therefore mgb(1+\cos\theta) = 1.5\,mgb$$

$$1+\cos\theta = 1.5$$

$$\cos\theta = 0.5$$

$$\theta = 60^\circ$$

$\therefore$ $BC$ makes an angle of $60^\circ$ with the upward vertical.

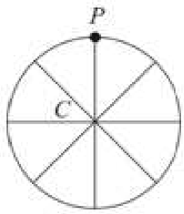
$k = 1.5$

P.E. gained = K.E. lost

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

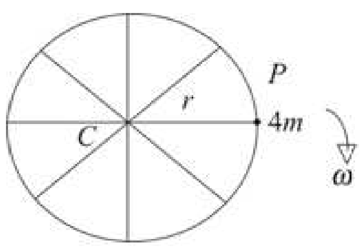
**Exercise A, Question 12**

**Question:**




A flywheel is made from a circular hoop of mass $6m$ and radius $r$ and four equally spaced rods, each of mass $m$ and length $2r$. A particle $P$ of mass $4m$ is attached to the hoop at the end of one rod. The loaded flywheel is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to the plane of the hoop through its centre, $C$. Initially the flywheel is at rest with $P$ vertically above $C$, as shown in the diagram. The wheel is then slightly disturbed and begins to rotate. Find, in terms of $r$ and g, the angular speed of the flywheel when $PC$ is horizontal.

**Solution:**




M.I. of flywheel and particle about given axis through $C$

$= 6mr^2 + 4 \times \frac{1}{3}mr^2 + 4mr^2$ ← The flywheel is a hoop and 4 rods.

$= \frac{34}{3}mr^2$

$\frac{1}{2}I\omega^2 = 4mgr$ ← K.E. gained = P.E. lost

$\frac{1}{2} \times \frac{34}{3}mr^2\omega^2 = 4mgr$

$\omega^2 = 4g \times \frac{6}{34r}$

$\omega = 2\sqrt{\frac{3g}{17r}}$

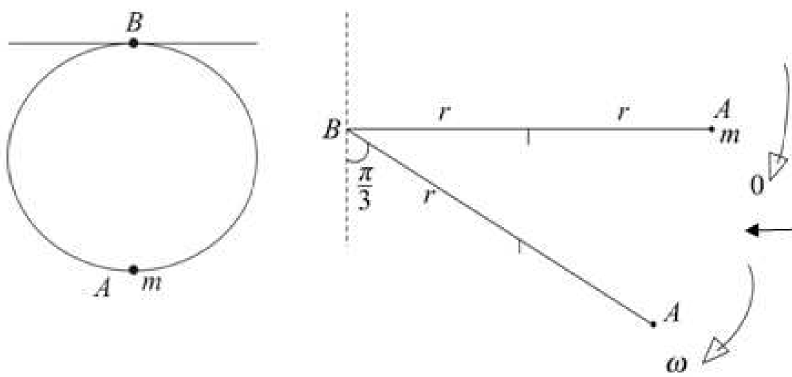
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 13**

**Question:**

A ring of mass $3m$ and radius $r$ has a particle of mass $m$ attached to it at the point $A$. The ring can rotate about a fixed smooth horizontal axis in the plane of the ring. The axis is tangential to the ring at the point $B$ where $AB$ is a diameter. The system is released from rest with $AB$ horizontal. Find the angular speed of the ring when $AB$ makes an angle $\frac{\pi}{3}$ with the downward vertical.

**Solution:**




The right-hand diagram is the side view.

M.I. of ring about perpendicular axis through centre $= 3mr^2$ ← From the formula book.

M.I. of ring about axis along a diameter $= \frac{1}{2} \times 3mr^2$ ← Perpendicular axes theorem.

M.I. of ring about tangential axis $= \frac{3}{2}mr^2 + 3mr^2$ ← Parallel axes theorem.

$\therefore$ M.I. of ring and particle about the given

$$\text{axis} = \frac{9}{2}mr^2 + m(2r)^2 = \frac{17}{2}mr^2$$

$$\frac{1}{2}I\omega^2 = 3mgr\cos\frac{\pi}{3} + mg(2r)\cos\frac{\pi}{3}$$ ← K.E. gained = P.E. lost

$$\frac{17}{4}mr^2\omega^2 = 3mgr \times \frac{1}{2} + 2mgr \times \frac{1}{2}$$

$$\frac{17}{4}r\omega^2 = \frac{5g}{2}$$

$$\omega^2 = \frac{10g}{17r}$$

$$\omega = \sqrt{\frac{10g}{17r}}$$

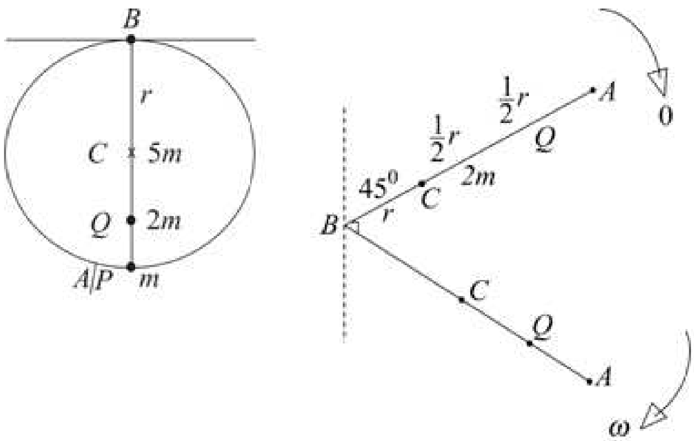
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise A, Question 14**

**Question:**

A uniform circular disc has mass $5m$ and radius $r$. A particle $P$ of mass $m$ is attached to the disc at point $A$ of its circumference. The centre of the disc is $C$. A second particle $Q$ of mass $2m$ is attached to the disc at the mid-point of $AC$. The disc is free to rotate about a fixed smooth horizontal axis in the plane of the disc. The axis is tangential to the disc at point $B$, where $AB$ is a diameter. The disc is released from rest with $AB$ at an angle $45°$ with the upward vertical. When $AB$ is at an angle $45°$ with the downward vertical the angular speed of the disc is $\omega$. Show that $\omega^2 = \dfrac{80g\sqrt{2}}{59r}$.

**Solution:**




M.I. of disc about perpendicular axis through $C$

$= \frac{1}{2} \times 5mr^2$ ← From the formula book.

M.I. of disc about a diameter $= \frac{1}{4} \times 5mr^2$ ← Perpendicular axes theorem.

M.I. of disc about a tangential axis

$= \frac{5}{4}mr^2 + 5mr^2 = \frac{25}{4}mr^2$ ← Parallel axes theorem.

M.I. of loaded disc about the given axis

$$= \frac{25mr^2}{4} + 2m\left(\frac{3r}{2}\right)^2 + m(2r)^2$$

$$= \frac{59mr^2}{4}$$

P.E. lost $= 5mgr\sqrt{2} + 2mg \times \frac{3r}{2}\sqrt{2} + mg \times 2r\sqrt{2}$

$= 10mgr\sqrt{2}$

$\therefore \frac{1}{2} \times \frac{59mr^2}{4}\omega^2 = 10mgr\sqrt{2}$ ← K.E. gained = P.E. lost

$$\omega^2 = \frac{80g\sqrt{2}}{59r}$$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 1**

**Question:**

A uniform rod of length $4a$ and mass $m$ is free to rotate in a horizontal plane about a fixed smooth vertical axis through its centre. A horizontal force of constant magnitude is applied to a free end of the rod in a direction perpendicular to the rod. The rod rotates with angular acceleration $12\frac{g}{a}$. Find the magnitude of the force.

**Solution:**


M.I. of rod $= \frac{1}{3}m \times (2a)^2 = \frac{4ma^2}{3}$

From the formula book.

$L = I\ddot{\theta}$

$F \times 2a = \frac{4ma^2}{3} \times \frac{12g}{a}$

$L$ is the moment of the force about the axis.

$F = 8mg$

The force has magnitude $8mg$.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 2**

**Question:**

A uniform disc of radius 0.5 m and mass 2.4 kg is free to rotate in a horizontal plane about a fixed smooth vertical axis through its centre. A horizontal force of constant magnitude 10 N is applied at point $A$ on the rim of the disc in the direction of the tangent to the disc at $A$.

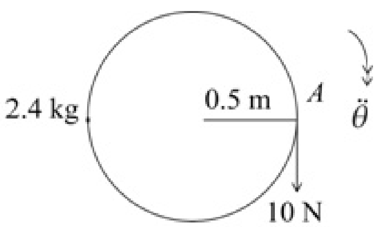
**a** Calculate the angular acceleration of the disc.

The disc starts from rest at time $t = 0$. Calculate

**b** the angular speed when $t = 2$,

**c** the angle the disc turns through in the first 2 s of the motion.

**Solution:**




**a** M.I. of disc $= \frac{1}{2} \times 2.4 \times 0.5^2$

$= 0.3 \text{ kg m}^2$

$10 \times 0.5 = 0.3\ddot{\theta}$

$\ddot{\theta} = \frac{10 \times 0.5}{0.3} = \frac{50}{3}$

The angular acceleration is $16\frac{2}{3}$ rad s$^{-2}$.

From the formula book.

Using $L = I\ddot{\theta}$

**b** $t = 2$ $\quad \omega_1 = \omega_0 + \alpha t$

$\omega_0 = 0$

$\alpha = \frac{50}{3}$ $\quad \omega_1 = 0 + \frac{50}{3} \times 2 = \frac{100}{3}$

The angular speed is $33\frac{1}{3}$ rad s$^{-1}$.

**c** $t = 2$ $\quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\omega_0 = 0$ $\quad \theta = 0 + \frac{1}{2} \times \frac{50}{3} \times 2^2$

$\alpha = \frac{50}{3}$ $\quad \theta = \frac{100}{3}$

The disc has turned through $33\frac{1}{3}$ rad.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 3**

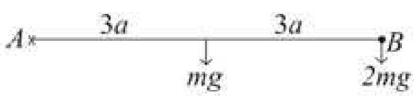
**Question:**

A uniform rod $AB$ of mass $m$ and length $6a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to $AB$ at $A$. A particle of mass $2m$ is attached to the rod at $B$. The loaded rod is released from rest with $AB$ horizontal. Find

**a** the initial angular acceleration of the rod,

**b** the angular acceleration when $AB$ makes an angle $\frac{\pi}{3}$ with the downward vertical.

**Solution:**




**a** M.I. of rod and particle about the given axis through $A = \frac{4}{3}m \times (3a)^2 + 2m \times (6a)^2$

$= 84ma^2$

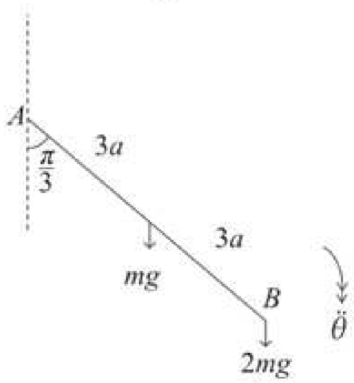
When the rod is released:

$mg \times 3a + 2mg \times 6a = 84ma^2\ddot{\theta}$ ← Using $L = I\ddot{\theta}$

$\ddot{\theta} = \frac{15g}{84a} = \frac{5g}{28a}$

The initial angular acceleration is $\frac{5g}{28a}$.

**b**




$mg \times 3a \sin\frac{\pi}{3} + 2mg \times 6a \sin\frac{\pi}{3} = 84ma^2\ddot{\theta}$ ← Using $L = I\ddot{\theta}$

$15g\frac{\sqrt{3}}{2} = 84a\ddot{\theta}$

$\ddot{\theta} = \frac{15g}{84a} \times \frac{\sqrt{3}}{2} = \frac{5g\sqrt{3}}{56}$

The angular acceleration is $\frac{5g\sqrt{3}}{56a}$.

# Solutionbank M5

## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 4**

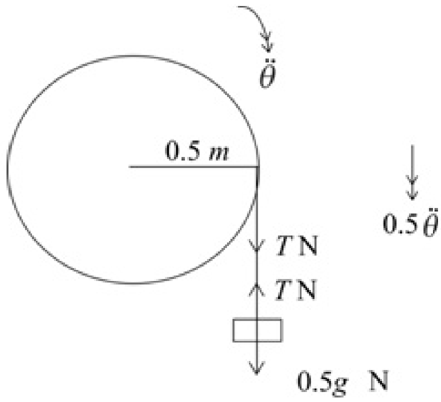
**Question:**

A pulley wheel of mass 2 kg and radius 0.5 m has one end of a rope attached to a point of the rim of the wheel. The rope is wound several times around the wheel. A fixed smooth horizontal axis passes through the centre of the wheel. A brick of mass 0.5 kg is attached to the free end of the rope. Initially the system is held at rest with the brick hanging freely with the rope taut. The system is then released and the wheel begins to rotate in a vertical plane perpendicular to the axis. The pulley wheel can be modelled as a uniform circular disc, the rope as a light inextensible string and the brick as a particle. Calculate

**a** the tension in the rope,

**b** the distance the brick falls in the first second after the system is released.

**Solution:**




The (linear) acceleration of the brick is $r\ddot{\theta}$.

**a** For the brick:

$0.5g - T = 0.5 \times 0.5\ddot{\theta}$ ①

Using $F = ma$ with $m = 0.5$ kg and $a = 0.5\ddot{\theta}$

For the wheel:

$\text{M.I.} = \frac{1}{2} \times 2 \times 0.5^2 = 0.25 \text{ kg m}^2$

$\therefore T \times 0.5 = 0.25\ddot{\theta}$

$T = 0.5\ddot{\theta}$ ②

Using $L = I\ddot{\theta}$

Substitute in ①:

$0.5g - T = 0.5T$

$\therefore T = \frac{1}{3}g$

The tension is $\frac{1}{3}g$ N (or 3.27 N)

**b** From ② $\ddot{\theta} = 2T = \frac{2}{3}g$

For the brick:

$a = 0.5\ddot{\theta} = \frac{1}{3}g$

Find the angular acceleration so you can use the constant acceleration equations.

$u = 0$

$t = 1$

$s = ut + \frac{1}{2}at^2$

$s = 0 + \frac{1}{2} \times \frac{g}{3} \times 1^2$

$s = 1.633$

The brick falls 1.63 m (3 s.f.) in the first second.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 5**

**Question:**

A uniform rod $AB$ of mass $m$ and length $4a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to $AB$ through the point $D$ of the rod where $AD = a$. The rod is released from rest with $AB$ horizontal. Calculate the magnitude of the force exerted on the axis

**a** when $AB$ is vertical with $A$ above $D$

**b** when $AB$ makes an angle of 45° with the downward vertical.

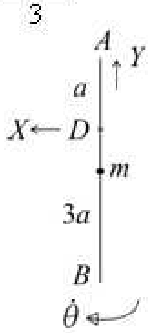
**Solution:**


M.I. of rod about given axis through $D$

$= \frac{1}{3}m \times (2a)^2 + ma^2$ ← Using the parallel axes theorem.

$= \frac{7ma^2}{3}$

**a**




You must find $\dot{\theta}$ and $\ddot{\theta}$ when the rod is vertical.

$\frac{1}{2}I\dot{\theta}^2 = mga$

$\frac{1}{2} \times \frac{7ma^2}{3}\dot{\theta}^2 = mga$ ← K.E. gained = P.E. lost

$\dot{\theta}^2 = \frac{6g}{7a}$

$\ddot{\theta} = 0$ (from $L = I\ddot{\theta}$) ← When the rod is vertical there is no force with a non-zero moment about the axis.

Consider the motion of a particle of mass $m$ at the centre of mass of the rod.

$Y - mg = ma\dot{\theta}^2$

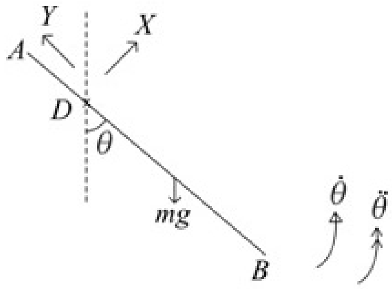
$Y = mg + ma \times \frac{6g}{7a}$ ← The particle is moving in a circle, radius $a$.

$Y = \frac{13}{7}mg$

$\ddot{\theta} = 0 \Rightarrow X = 0$

$\therefore$ Magnitude is $\frac{13mg}{7}$

**b**




$\dot{\theta}$ and $\ddot{\theta}$ are in the direction of increasing $\theta$.

At $\theta$ to the downward vertical:

$$\frac{1}{2}\times\frac{7ma^2}{3}\dot{\theta}^2 = mga\cos\theta$$

K.E. gained = P.E. last

$$\dot{\theta}^2 = \frac{6g}{7a}\cos\theta$$

When $\theta = 45°$ $\quad \dot{\theta}^2 = \frac{6g}{7a}\times\frac{1}{\sqrt{2}}$

$$2\ddot{\theta} = -\frac{6g}{7a}\sin\theta$$

You can differentiate $\dot{\theta}^2$ with respect to $\theta$ to obtain $\ddot{\theta}$. Alternatively, you can use $L = I\ddot{\theta}$

When $\theta = 45°$ $\ddot{\theta} = -\frac{3g}{7a}\times\frac{1}{\sqrt{2}}$

For the particle at the centre of mass of the rod:
Parallel to the rod:

$$Y - mg\cos\theta = ma\dot{\theta}^2$$

$$Y = mg\times\frac{1}{\sqrt{2}} + ma\times\frac{6g}{7a}\times\frac{1}{\sqrt{2}}$$

$$Y = \frac{13mg}{7\sqrt{2}}$$

Perpendicular to the rod:

$$X - mg\sin\theta = ma\ddot{\theta}$$

$$X = mg\times\frac{1}{\sqrt{2}} - ma\times\frac{3g}{7a\sqrt{2}}$$

$$X = \frac{4mg}{7\sqrt{2}}$$

$\therefore$ Magnitude of the force $= \sqrt{(X^2+Y^2)}$

The magnitude is the same for the force on the axis and the force on the particle.

$$= \frac{mg}{7\sqrt{2}}\sqrt{\left(13^2+4^2\right)}$$

$$= \frac{mg}{7}\sqrt{\frac{185}{2}}$$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 6**

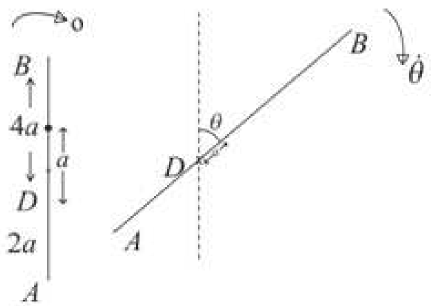
**Question:**

A uniform rod $AB$ of mass $m$ and length $6a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to $AB$ through the point $D$ of the rod where $AD = 2a$. The rod is initially at rest with $A$ vertically below $D$ but is then slightly disturbed and starts to rotate. Find

**a** the angular speed when $AB$ has turned through an angle $\theta$,

**b** the magnitude of the force on the axis when the rod is vertical with $B$ below $D$.

**Solution:**




**a** M.I. of rod about given axis through $D$

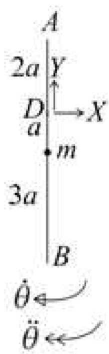
$= \frac{1}{3}m \times (3a)^2 + ma^2 = 4ma^2$ ← Use the parallel axes theorem.

$\frac{1}{2}I\dot{\theta}^2 = mga(1-\cos\theta)$ ← K.E. gained = P.E. lost

$2ma^2\dot{\theta}^2 = mga(1-\cos\theta)$

$\dot{\theta} = \sqrt{\left[\frac{g}{2a}(1-\cos\theta\right]}$

**b**




Using $L = I\ddot{\theta}$

There is no horizontal force other than $x \Rightarrow \ddot{\theta} = 0$

Consider the motion of a particle of mass $m$ at the centre of mass of the rod: ← Alternatively, when $B$ is vertically below $D$ the angular speed is maximum, so $\ddot{\theta} = 0$

Along the rod: $Y - mg = ma\dot{\theta}^2$

$Y - mg = ma \times \frac{g}{2a}[1-(-1)]$ ← $\theta = 180° \Rightarrow \cos\theta = -1$

$Y - mg = mg$

$Y = 2mg$

Perpendicular to the rod:

$\ddot{\theta} = 0$

$\therefore X = 0$

The magnitude of the force is $2mg$. ← The magnitude is the same for the force on the axis and the force on the particle.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 7**

**Question:**

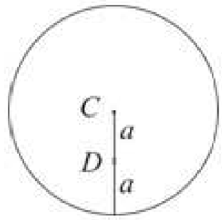
A uniform circular disc of mass $m$ and radius $2a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to the disc through a point, $D$, which is at a distance $a$ from the centre of the disc, $C$. The disc is initially at rest with $C$ vertically above $D$. The disc is then slightly disturbed and begins to rotate. Find the magnitude of the force on the axis

**a** when $CD$ is horizontal

**b** when $CD$ is vertical with $C$ below $D$.

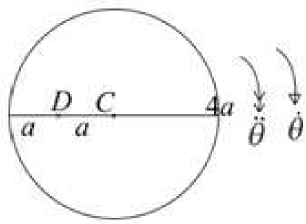
**Solution:**

**a**




M.I. of disc about axis through $C = \frac{1}{2}m(2a)^2 = 2ma^2$ ← From the formula book.

M.I. of disc about axis through $D = 2ma^2 + ma^2$
$= 3ma^2$ ← By the parallel axes theorem.




Energy from rest to $CD$ horizontal

$$\frac{1}{2}I\dot{\theta}^2 = mga$$ ← K.E. gained = P.E. lost

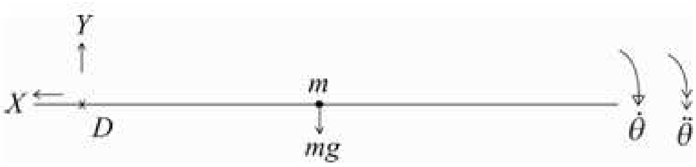
$$\frac{3ma^2}{2}\dot{\theta}^2 = mga$$

$$\dot{\theta}^2 = \frac{2g}{3a}$$

$$L = I\ddot{\theta}$$ ← Equation of rotational motion

$$mga = 3ma^2\ddot{\theta}$$

$$\ddot{\theta} = \frac{g}{3a}$$




Consider the motion of a particle of mass $m$ at the centre of mass of the disc:

Horizontally:

$$X = ma\dot{\theta}^2$$ ← Use $F = ma$

$$X = ma \times \frac{2g}{3a}$$

$$X = \frac{2mg}{3}$$

Vertically:

$$mg - Y = ma\ddot{\theta}$$

$$Y = mg - ma \times \frac{g}{3a}$$

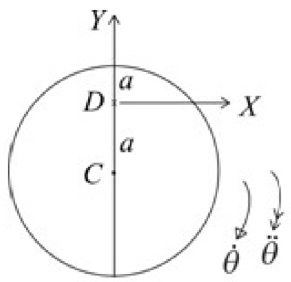
$$Y = \frac{2mg}{3}$$

$$\therefore \text{ Magnitude} = \sqrt{\left(\frac{2mg}{3}\right)^2 + \left(\frac{2mg}{3}\right)^2}$$

$$= \frac{2mg}{3}\sqrt{2}$$

Use $F = ma$

**b**




Energy from rest to $CD$ vertical:

$$\frac{1}{2}I\dot{\theta}^2 = 2mga$$

$$\frac{3ma^2}{2}\dot{\theta}^2 = 2mga$$

$$\dot{\theta}^2 = \frac{4g}{3a}$$

K.E. gained = PE lost

No horizontal force apart from $x \Rightarrow \ddot{\theta} = 0$

Use the equation of rotational motion

For a particle of mass $m$ at $C$

Vertically:

$$Y - mg = ma\dot{\theta}^2$$

$$Y = mg + \frac{4mg}{3} = \frac{7mg}{3}$$

Use $F = ma$

Horizontally: $X = -ma\ddot{\theta} = 0$

Use $F = ma$

$\therefore$ Magnitude is $\dfrac{7mg}{3}$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 8**

**Question:**

A uniform rod $AB$ of mass $m$ and length $2a$ is attached to a fixed smooth hinge at $A$. The rod is released from rest from a horizontal position and rotates in a vertical plane perpendicular to the hinge.

**a** Show that, when $AB$ has rotated through an angle $\theta$

$$2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$$

When $AB$ has rotated through an angle $\theta$, the force exerted by $AB$ on the axis is $F$.

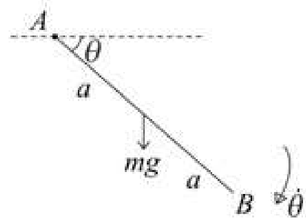
**b** Find the magnitudes of the components, parallel and perpendicular to $AB$, of $F$.

**c** Show that the horizontal component of $F$ is greatest when $\theta = \frac{\pi}{4}$.

**d** Find the vertical component of $F$ when $\theta = \frac{\pi}{4}$.

**Solution:**

**a**




M.I. of rod about axis through $A = \frac{4}{3}ma^2$ ← From the formula book.

Energy:

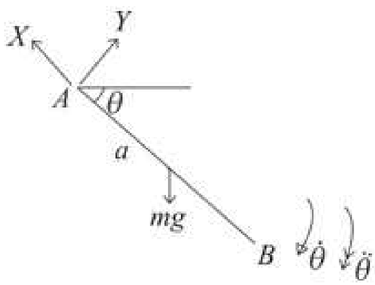
$$\frac{1}{2}\times\left(\frac{4}{3}ma^2\right)\dot{\theta}^2 = mga\sin\theta$$

← The rod starts from rest with $AB$ horizontal.

$$2a\dot{\theta}^2 = 3g\sin\theta$$

$$\text{or } 2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$$

**b**




← Consider the motion of a particle of mass $m$ at the centre of mass of the rod.

For the component parallel to $AB$:

$$X - mg\sin\theta = ma\dot{\theta}^2$$

← Use $F = ma$

$$X = mg\sin\theta + m\times\frac{3g}{2}\sin\theta$$

← Use the result from **a**.

$$X = \frac{5mg}{2}\sin\theta$$

For the component perpendicular to $AB$:

$$mg\cos\theta - Y = ma\ddot{\theta}$$

← Use $F = ma$

$$2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$$

$$4a\frac{d^2\theta}{dt^2} = 3g\cos\theta$$

← Differentiate the result from **a** with respect to $\theta$

$$a\ddot{\theta} = \frac{3g}{4}\cos\theta$$

$$\therefore Y = mg\cos\theta - \frac{3}{4}mg\cos\theta = \frac{1}{4}mg\cos\theta$$

**c** Horizontal component

$= X\cos\theta - Y\sin\theta$

$= \frac{5}{2}mg\sin\theta\cos\theta - \frac{1}{4}mg\sin\theta\cos\theta$

$= \frac{9mg}{4}\sin\theta\cos\theta$

$= \frac{9}{8}mg\sin 2\theta$

You can differentiate this to obtain the maximum but the trigonometric method is much simpler!

$\therefore$ Horizontal component is maximum when $\sin 2\theta = 1$

$\theta = \frac{\pi}{4}$

$\therefore$ Maximum when $\theta = \frac{\pi}{4}$

**d** Vertical component

$= X\sin\theta + Y\cos\theta$

$= \frac{5}{2}mg\sin^2\theta + \frac{1}{4}mg\cos^2\theta$

$\theta = \frac{\pi}{4}$

Vertical component

$= \frac{5}{2}mg\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{4}mg\left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{5mg}{4} + \frac{1}{8}mg$

$= \frac{11}{8}mg$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 9**

**Question:**

A uniform wire of mass $m$ and length $6a$ is bent to form a rectangle $ABCD$ with $AB = 2a$ . It is hung with corner $A$ over a fixed smooth horizontal nail. Initially it is held at rest with $AB$ horizontal and $D$ below $A$. The plane of the rectangle is perpendicular to the nail.

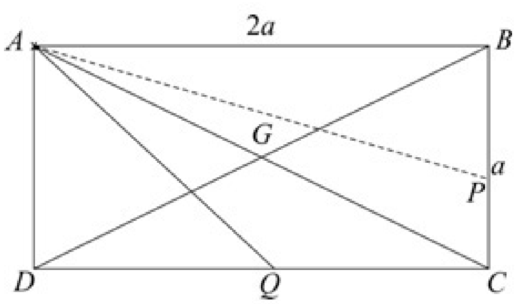
**a** Show that the moment of inertia of the framework about the nail is $2ma^2$.

**b** Show that the angular speed $\dot{\theta}$ of the wire when $AC$ is vertical is given by $\dot{\theta}^2 = \frac{g}{2a}(\sqrt{5}-1)$.

**c** Find the magnitude of the resultant force on the nail when $AC$ is vertical.

**Solution:**

**a**




$G$ is the centre of mass of the framework.

$P$ and $Q$ are the mid-points of $BC$ and $CD$ respectively.

M.I. of rectangle about nail

$$= \frac{4}{3}\times\left(\frac{1}{3}m\right)\times a^2 + \left\{\frac{1}{3}\times\frac{1}{6}m\left(\frac{1}{2}a\right)^2 + \frac{1}{6}m\left(4a^2+\frac{1}{4}a^2\right)\right\}$$

$$+\left\{\frac{1}{3}\times\left(\frac{1}{3}m\right)\times a^2 + \frac{1}{3}m\times 2a^2\right\} + \frac{4}{3}\times\left(\frac{1}{6}m\right)\left(\frac{a}{2}\right)^2$$

$$= \frac{4ma^2}{9} + \frac{ma^2}{72} + \frac{17ma^2}{24} + \frac{ma^2}{9} + \frac{2ma^2}{3} + \frac{ma^2}{18}$$

$$= 2ma^2$$

You must work from the centres of mass when using the parallel axes theorem for $BC$ and $CD$.

$AP^2 = 4a^2 + \frac{1}{4}a^2$

$AQ^2 = a^2 + a^2$

**b** Energy:

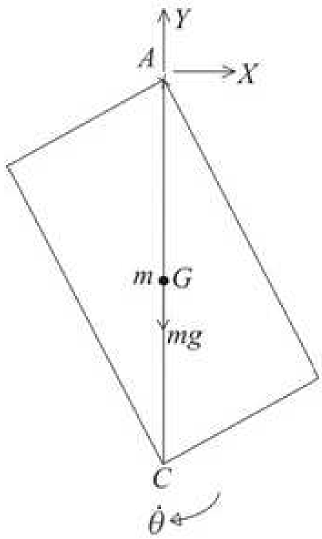
$$\frac{1}{2}\times 2ma^2\dot{\theta}^2 = mg\left(\frac{a}{2}\sqrt{5} - \frac{a}{2}\right)$$

$$2a\dot{\theta}^2 = g(\sqrt{5}-1)$$

$$\dot{\theta}^2 = \frac{g}{2a}(\sqrt{5}-1)$$

$AG^2 = a^2 + \frac{1}{4}a^2 = \frac{5a^2}{4}$

**c**




Consider the motion of a particle of mass $m$ placed at the centre of mass of the framework.

Vertically:

$Y - mg = m \times AG \times \dot{\theta}^2$

$Y - mg = \frac{ma}{2}\sqrt{5} \times \frac{g}{2a}(\sqrt{5} - 1)$

$AG = \frac{a}{2}\sqrt{5}$ (from **a**)

$Y = mg + \frac{mg}{4} \times 5 - \frac{mg}{4}\sqrt{5}$

$= \frac{9mg}{4} - \frac{mg\sqrt{5}}{4}$

Horizontally:

$-X = \frac{ma}{2}\sqrt{5}\ddot{\theta}$

$\dot{\theta}$ is maximum when $AC$ is vertical

Or use $L = I\ddot{\theta}$ with $L = 0$

$\Rightarrow \ddot{\theta} = 0$

$\therefore X = 0$

$\therefore$ The magnitude of the resultant force on the nail is $\frac{9mg}{4} - mg\frac{\sqrt{5}}{4}$ or

$\frac{mg}{4}(9 - \sqrt{5})$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise B, Question 10**

**Question:**

A uniform square lamina $ABCD$ of mass $m$ and side $4a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through $A$ perpendicular to $ABCD$. The lamina is hanging in equilibrium with $C$ below $A$ when it receives an impulse and begins to rotate with angular speed $\sqrt{\frac{3g}{a}}$.

**a** Show that the lamina will perform complete revolutions.

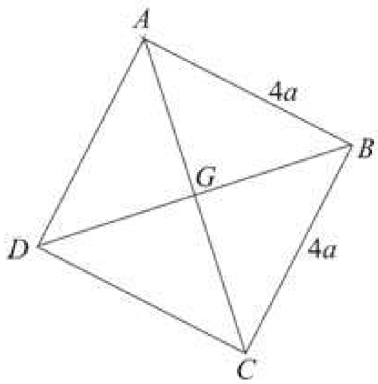
**b** Find the magnitude of the horizontal and vertical components of the force on the axis

**i** when $C$ is vertically above $A$,

**ii** when $AC$ is horizontal.

**Solution:**

**a**




G is the centre of mass of $ABCD$.
$AG^2 = (2a)^2 + (2a)^2$
$= 8a^2$

M.I. of lamina about axis through $A = \frac{1}{3}m\left(4a^2 + 4a^2\right) + m \times 8a^2$

$$= \frac{32ma^2}{3}$$

Energy:

$$\frac{1}{2} \times \frac{32ma^2}{3} \times \frac{3g}{a} - \frac{1}{2} \times \frac{32ma^2}{3}\omega^2 = mg \times 2a\sqrt{8}$$

$$16g - \frac{16a\omega^2}{3} = 4g\sqrt{2}$$

$\omega$ is the angular speed when $C$ is vertically above $A$.

For complete revolutions $\omega^2 \geq 0$

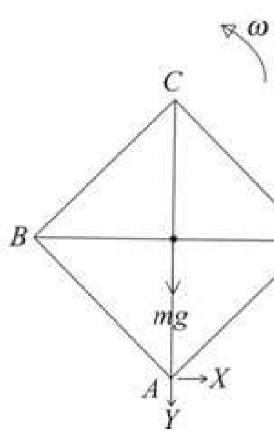
$$\frac{16a\omega^2}{3} = 16g - 4g\sqrt{2}$$

$$a\omega^2 = \frac{3}{4}g(4 - \sqrt{2}) > 0$$

$\therefore$ The lamina will perform complete revolutions.

As long as it is clear that $\omega^2 > 0$ there is no need to evaluate $\frac{3}{4}g(4 - \sqrt{2})$

**b i**




Consider the motion of a particle of mass $m$ at the centre of mass of the lamina.

When $C$ is vertically above $A$

$$\omega^2 = \frac{3g}{4a}\left(4 - \sqrt{2}\right)$$

From **a**.

Vertically:

$mg + Y = m \times 2a\sqrt{2}\omega^2$

$Y = 2ma\sqrt{2} \times \frac{3g}{4a}(4-\sqrt{2}) - mg$

$Y = 6mg\sqrt{2} - 4mg$

Use $F = ma$

Horizontally:

when $AC$ is vertical angular speed is a minimum.

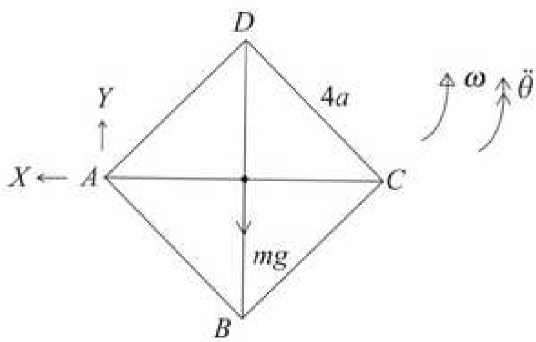
$\therefore \ddot{\theta} = 0$

$\therefore$ horizontal component of force $= 0$

The horizontal component is zero and the vertical component is $2mg(3\sqrt{2}-2)$.

Or you can use $L = I\ddot{\theta}$ with $L = 0$

**ii**




Energy (to $AC$ being horizontal):

$$\frac{1}{2} \times \frac{32ma^2}{3} \times \frac{3g}{a} - \frac{1}{2} \times \frac{32ma^2}{3}\omega_1^2 = mg \times a\sqrt{8}$$

$$16g - \frac{16}{3}a\omega_1^2 = g\sqrt{8}$$

$$\omega_1^2 = \frac{3g}{16a}(16-\sqrt{8})$$

Horizontally:

$X = ma\sqrt{8} \times \frac{3g}{16a}(16-\sqrt{8})$

Use $F = ma$.

$= \frac{3mg}{2}(4\sqrt{2}-1)$

Any equivalent form is acceptable.

Vertically:

$Y - mg = ma\sqrt{8}\ddot{\theta}$

Use $F = ma$.

$mga\sqrt{8} = -\frac{32ma^2}{3}\ddot{\theta}$

Use $L = I\ddot{\theta}$.

$\therefore ma\ddot{\theta} = -mg\sqrt{8} \times \frac{3}{32}$

$\therefore Y = mg - mg\sqrt{8} \times \frac{3}{32}\sqrt{8} = \frac{mg}{4}$

The horizontal component has magnitude $\frac{3mg}{2}(4\sqrt{2}-1)$ and the vertical component has magnitude $\frac{1}{4}mg$.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

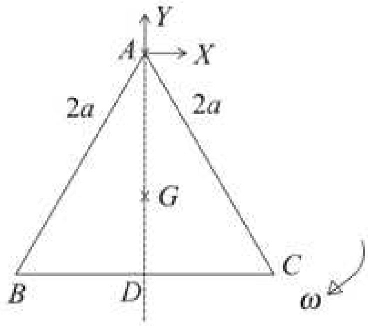
**Exercise B, Question 11**

**Question:**

Three equal uniform rods, each of mass $m$ and length $2a$, are joined to form an equilateral triangle $ABC$. The triangular frame can rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to $ABC$ through $A$. The mid-point of $BC$ is $D$. The frame is released from rest with $AD$ horizontal and $C$ below $AB$. Find the magnitude of the force on the axis when $AD$ is vertical.

[You may assume that the centre of mass of the triangle is at $G$ where $G$ divides $AD$ in the ratio 2 : 1.]

**Solution:**




$G$ is the centre of mass of the framework. $AG : GD = 2 : 1$

M.I. of triangle about axis through $A$

$= 2 \times \frac{4}{3} ma^2 + \frac{1}{3} ma^2 + m \times 3a^2$

$= \frac{8ma^2}{3} + \frac{ma^2}{3} + 3ma^2$

$= 6ma^2$

$AD^2 = 4a^2 - a^2$
$= 3a^2$

From the formula book and using the parallel axes theorem.

Energy:

$\frac{1}{2} \times 6ma^2\omega^2 = 3mg \times \frac{2}{3} \times a\sqrt{3}$

$ma\omega^2 = \frac{2mg}{3}\sqrt{3}$

$AD$ was horizontal at the start.

When $AD$ is vertical:

vertically:

$Y - 3mg = 3m \times \frac{2a}{3}\sqrt{3}\omega^2$

$Y = 3mg + 2\sqrt{3} \times \frac{2}{3} mg\sqrt{3}$

$Y = 7mg$

Consider the motion of a particle of mass $3m$ placed at $G$.

Use $F = ma$.

$AG = \frac{2}{3} AD = \frac{2a}{3}\sqrt{3}$

When $AD$ is vertical, angular speed is maximum.

$\therefore \ddot{\theta} = 0$

$\therefore$ no horizontal component

$\therefore$ The force on the axis has magnitude $7mg$.

Or use $L = I\ddot{\theta}$ with $L = 0$

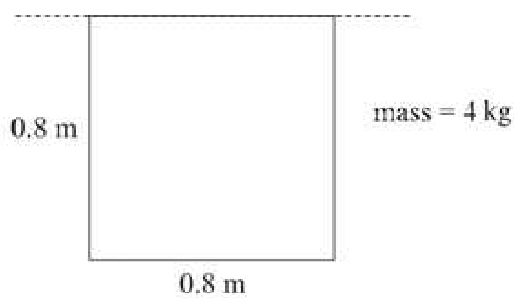
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 1**

**Question:**

A uniform square lamina of side 0.8 m and mass 4 kg is free to rotate about a fixed smooth axis which coincides with one of its sides. Calculate the gain of angular momentum when the angular speed of the lamina is increased from $2 \text{ rad s}^{-1}$ to $5 \text{ rad s}^{-1}$.

**Solution:**




M.I. of lamina about given axis $= \frac{4}{3} \times 4 \times 0.4^2$ ← From the formula book.

Gain in angular momentum

$= I\omega_1 - I\omega_0$

$= \frac{4}{3} \times 4 \times 0.4^2 (5 - 2)$

$= 2.56 \text{ Nms}$

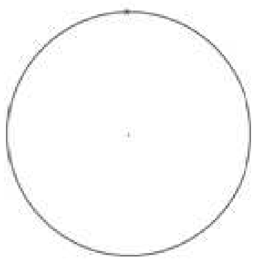
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 2**

**Question:**

A uniform hoop of mass 1.2 kg and radius 1.5 m is rotating at a constant angular speed of 6 rad $s^{-1}$ about a fixed smooth horizontal axis through a point of the circumference of the hoop, perpendicular to the plane of the hoop. Calculate the angular momentum of the hoop.

**Solution:**



M.I. of hoop about perpendicular axis through its centre $= mr^2$ ← From the formula book.

$= 1.2 \times 1.5^2$ kg $m^2$

M.I. of hoop about given axis $= 1.2 \times 1.5^2 + 1.2 \times 1.5^2$ ← Use the parallel axes theorem.

$= 5.4$ kg$m^2$

Angular momentum

$= 5.4 \times 6$

$= 32.4$ Nms ← Angular momentum $= I\omega$

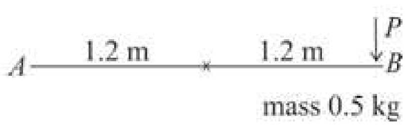
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 3**

**Question:**

A uniform rod $AB$ of length 2.4 m and mass 0.5 kg is rotating in a horizontal plane at 6 rad $s^{-1}$ about a fixed smooth vertical axis through its centre. A retarding force of constant magnitude $P$ newtons is applied at $B$ in a direction perpendicular to $AB$ in the plane of the motion. The rod is brought to rest in 5 seconds. Calculate the value of $P$.

**Solution:**




M.I. of rod about vertical axis through centre $= \frac{1}{3}ml^2$ ← From the formula book.

$$= \frac{1}{3} \times 0.5 \times 1.2^2 \text{kgm}^2$$

Angular momentum lost

$$= \frac{1}{3} \times 0.5 \times 1.2^2 \times 6 \text{Nms}$$

$$\therefore P \times 5 \times 1.2 = \frac{1}{3} \times 0.5 \times 1.2^2 \times 6$$

$$P = \frac{1}{3} \times \frac{0.5 \times 1.2^2 \times 6}{5 \times 1.2}$$

$$P = 0.24$$

Moment of impulse = change in angular momentum

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 4**

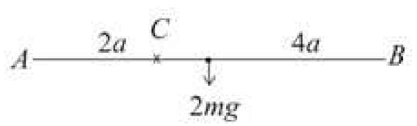
**Question:**

A uniform rod $AB$ of mass $2m$ and length $6a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through the point $C$ of the rod where $AC = 2a$. The rod is released from rest with $AB$ horizontal. When the rod is vertical with $B$ below $C$, the end $B$ strikes a stationary particle of mass $m$. The particle adheres to the rod.

**a** Show that the angular speed of the rod immediately after the impact is $\frac{1}{3}\sqrt{\frac{g}{2a}}$.

**b** Calculate the angle between the rod and the downward vertical when the rod first comes to instantaneous rest.

**Solution:**




M.I. of rod about horizontal axis through $C$

$= \frac{1}{3}(2m)\times(3a)^2 + 2ma^2$

$= 8ma^2$

From the formula book and using the parallel axes theorem.

**a** Energy from release to impact:

$\frac{1}{2}I\dot{\theta}^2 = 2mga$

K.E. gained = P.E. lost

$4ma^2\dot{\theta}^2 = 2mga$

$\dot{\theta}^2 = \frac{2mga}{4ma^2} = \frac{g}{2a}$

For the impact:

$I\dot{\theta} = \left[I + m(4a)^2\right]\omega$

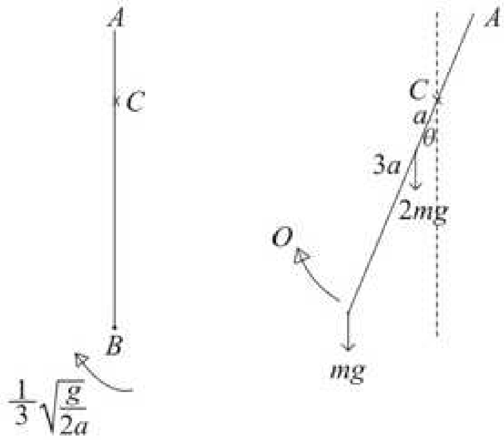
$\omega$ is the angular speed after the impact.

$8ma^2\sqrt{\frac{g}{2a}} = \left(8ma^2 + 16ma^2\right)\omega$

$8ma^2\sqrt{\frac{g}{2a}} = 24ma^2\omega$

$\omega = \frac{1}{3}\sqrt{\frac{g}{2a}}$

**b**




Energy to the highest point:

$$\frac{1}{2}I'\omega^2 = 2mg(a - a\cos\theta) + mg(4a - 4a\cos\theta)$$

$I'$ is the M.I. of the rod and the particle.

$$\frac{1}{2}\times 24ma^2\left(\frac{1}{3}\sqrt{\frac{g}{2a}}\right)^2 = 6mga(1-\cos\theta)$$

$I' = 24ma^2$ from **a**

$$12a\times\frac{1}{9}\times\frac{g}{2a} = 6g(1-\cos\theta)$$

$$\frac{6}{9} = 6(1-\cos\theta)$$

$$\cos\theta = 1-\frac{1}{9} = \frac{8}{9}$$

$$\theta = 27.26\ldots$$

The angle between the rod and the downward vertical is $27.3°$ (3 s.f.)

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 5**

**Question:**

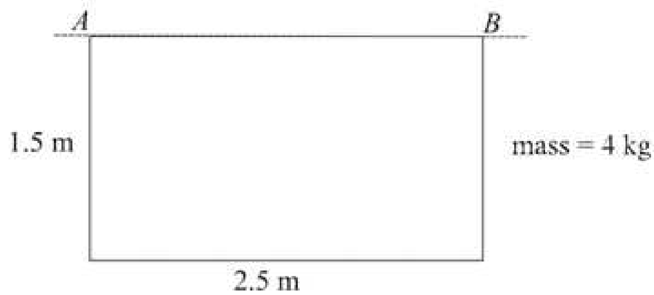
A rectangular sign is hanging outside a shop. The sign has mass 4 kg and measures 1.5 m by 2.5 m. It is free to rotate about a fixed smooth horizontal axis which coincides with a long side of the sign. The sign is hanging vertically at rest when it receives an impulse, perpendicular to its plane, at its centre of mass. The sign first comes to rest when it is horizontal. Calculate

**a** the initial angular speed of the sign,

**b** the magnitude of the impulse.

(You may assume that the sign can be modelled as a uniform rectangular lamina.)

**Solution:**




M.I. of sign about axis along $AB = \frac{4}{3} \times 4 \times \left(\frac{1.5}{2}\right)^2 = 3 \text{ kg m}^2$ ← From the formula book.

**a** Energy from just after the impact until the sign is horizontal.

$\frac{1}{2} I\omega^2 = 4g \times 0.75$ ← $\omega$ is the initial angular speed

$\frac{1}{2} \times 3\omega^2 = 4g \times 0.75$

$\omega^2 = \frac{8g \times 0.75}{3}$

$\omega = 4.427\ldots$

The initial angular speed is $4.4 \text{ rad s}^{-1}$ (2 s.f.)

**b** For the impact:

Moment of impulse = change in angular momentum

$J \times 0.75 = 3 \times 4.427$ ← $J$ is the magnitude of the impulse.

$J = \frac{3 \times 4.427}{0.75}$

$J = 17.7\ldots$

The magnitude of the impulse is 18 N (2 s.f.)

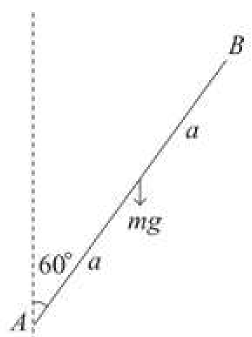
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 6**

**Question:**

A uniform rod $AB$ of mass $m$ and length $2a$ is freely hinged at $A$. The rod is released from rest with $AB$ at $60°$ with the upward vertical through $A$. When $AB$ is horizontal it hits a small fixed peg at point $C$ where $AC = 1.5a$. The angular speed of the rod immediately after the impact is half its speed immediately before the impact. Find the impulse exerted by the peg on the rod.

**Solution:**




M.I. of rod about axis through $A = \frac{4}{3}ma^2$ ← From the formula book.

Energy from release to horizontal:

$$\frac{1}{2}I\dot{\theta}^2 = mga\cos 60$$ ← gain of K.E. = loss of P.E.

$$\frac{2}{3}ma^2\dot{\theta}^2 = mga \times \frac{1}{2}$$

$$\dot{\theta}^2 = \frac{3g}{4a}$$


For the impact

$$1.5aJ = I\sqrt{\frac{3g}{4a}} + I \times \frac{1}{2}\sqrt{\frac{3g}{4a}}$$ ← $J$ is the magnitude of the impulse. The direction of rotation is reversed.

$$1.5aJ = \left(\frac{4}{3}ma^2 + \frac{1}{2}\times\frac{4}{3}ma^2\right)\sqrt{\frac{3g}{4a}}$$

$$1.5a\,J = 2ma^2\sqrt{\frac{3g}{4a}}$$

$$J = \frac{2ma}{2}\sqrt{\frac{3g}{a}} \times \frac{1}{1.5} = \frac{2m}{3}\sqrt{3ga}$$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 7**

**Question:**

A uniform rod $AB$ of mass $m$ and length $2a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through $A$. When the rod is hanging at rest with $B$ vertically below $A$, the end $B$ receives an impulse of magnitude $J$ in a direction perpendicular to the axis of rotation.

**a** Show that, for the rod to rotate in a complete circle,

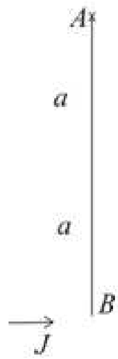
$$J \geq 2m\sqrt{\frac{ga}{3}}$$

Given that $J = \frac{2m}{3}\sqrt{\frac{ga}{3}}$

**b** find the angle the rod turns through before first coming to instantaneous rest.

**Solution:**

**a**




M.I. of rod about axis through $A = \frac{4}{3}ma^2$ ← From the formula book.

For the impact:

$2aJ = I\omega$

$$\omega = 2aJ \times \frac{3}{4ma^2} = \frac{3J}{2ma}$$

Energy from impact to $B$ vertically above $A$:

$$\frac{1}{2}I\omega^2 - \frac{1}{2}I\dot{\theta}^2 = mg \times 2a$$ ← Loss of K.E. = gain of P.E.

$$\frac{2ma^2}{3}\left(\frac{3J}{2ma}\right)^2 - \frac{2ma^2}{3}\dot{\theta}^2 = 2mga$$

For complete circles $\dot{\theta}^2 \geq 0$

$$\therefore \frac{2ma^2}{3} \times \frac{9J^2}{4m^2a^2} - 2mga \geq 0$$

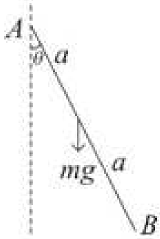
$$J^2 \geq 2mga \times \frac{2m^2a^2}{3ma^2}$$

$$J^2 \geq \frac{4m^2ga}{3}$$

$$J \geq 2m\sqrt{\frac{ga}{3}}$$

**b** $J = \frac{2m}{3}\sqrt{\frac{ga}{3}}$

$\therefore$ speed just after the impact $= \frac{3J}{2ma} = \frac{3}{2ma} \times \frac{2m}{3}\sqrt{\frac{ga}{3}} = \sqrt{\frac{g}{3a}}$ ← Use your result from **a**.




Energy from lowest to highest point:

$$\frac{1}{2}\left(\frac{4ma^2}{3}\right)\left(\sqrt{\frac{g}{3a}}\right)^2 = mga(1-\cos\theta)$$ ← K.E. lost = P.E. gained

$$\frac{2ma^2}{3} \times \frac{g}{3a} = mga(1-\cos\theta)$$

$$\frac{2}{9} = 1-\cos\theta$$

$$\cos\theta = \frac{7}{9}$$

$$\theta = 38.94\ldots$$

The rod turns through $38.9°$ (3 s.f.)

# Solutionbank M5

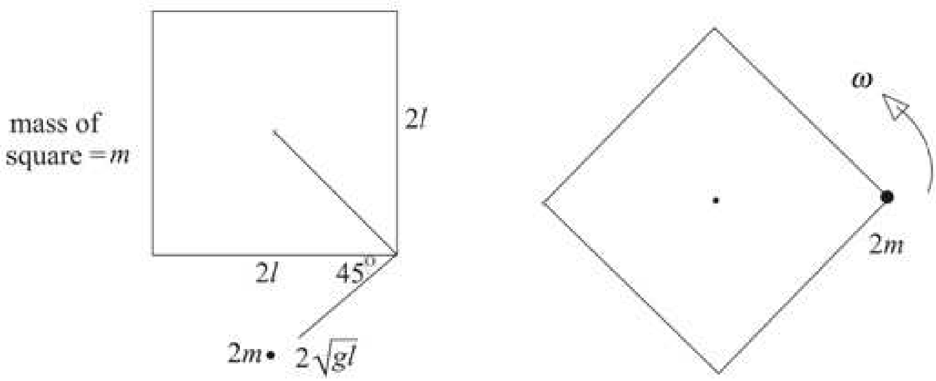
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 8**

**Question:**

A uniform square lamina of mass $m$ and side $2l$ is free to rotate in a horizontal plane about a fixed smooth vertical axis through the centre of the lamina. Initially the lamina is at rest. A particle of mass $2m$ is moving in the plane of the lamina towards the lamina with speed $2\sqrt{gl}$ and in a direction at 45° to a side. The particle strikes and adheres to the lamina at a corner. Find the angular speed with which the lamina begins to turn.

**Solution:**




M.I. of square lamina about perpendicular axis through centre

$= \frac{1}{3}m(l^2 + l^2)$ ← From formula book.

$= \frac{2}{3}ml^2$

Conservation of angular momentum:

$2m \times 2\sqrt{gl} \times l\sqrt{2} = \left[\frac{2ml^2}{3} + 2m(l\sqrt{2})^2\right]\omega$ ← Diagonal of the square is $2l\sqrt{2}$. ← Remember to include the M.I. of the particle.

$4ml\sqrt{2gl} = \frac{14ml^2}{3}\omega$

$\omega = \frac{12\sqrt{2gl}}{14l} = \frac{6}{7}\sqrt{\frac{2g}{l}}$ ← Equivalent forms of this answer are acceptable.

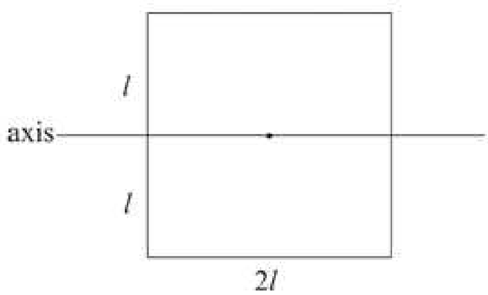
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 9**

**Question:**

A uniform square lamina of mass $m$ and side $2l$ is rotating with angular speed $\sqrt{\frac{6g}{l}}$ about a fixed smooth horizontal axis through the centre of the lamina parallel to one side of the lamina. A particle of mass $2m$ is held at a height $12l$ above the level of the axis of rotation of the lamina. The particle is released from rest and hits the lamina at an instant when the lamina is horizontal. The particle adheres to the lamina at the mid-point of a side which is moving downwards at the instant of impact. Find the angular speed of the lamina immediately after the impact.

**Solution:**




M.I. of lamina $= \frac{1}{3}ml^2$ ← From the formula book.

Particle falling freely under gravity:

$s = 12l \quad u = 0$

$a = g \quad v^2 = u^2 + 2as$

$v^2 = 24gl$

For the impact:

Conservation of angular momentum:

$$2m\sqrt{24gl} \times l + \frac{1}{3}ml^2\left(\sqrt{\frac{6g}{l}}\right) = \left(2ml^2 + \frac{1}{3}ml^2\right)\omega$$

← The particle is a distance $l$ from the axis.

$$2 \times 2\sqrt{6gl} + \frac{1}{3}\sqrt{6gl} = \frac{7}{3}l\omega$$

$$\frac{13\sqrt{6gl}}{3} = \frac{7}{3}l\omega$$

$$\omega = \frac{13}{7}\sqrt{\frac{6g}{l}}$$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

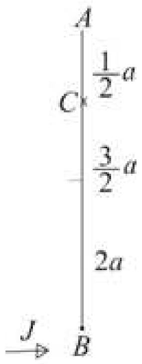
**Exercise C, Question 10**

**Question:**

A uniform rod $AB$ of mass $m$ and length $4a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through point $C$ of the rod, where $AC = \frac{1}{2}a$. When the rod is hanging at rest with $B$ vertically below $A$, the end $B$ receives an impulse of magnitude $J$ in a direction perpendicular to the axis of rotation. The impulse is sufficient to cause the rod to move in a complete circle. Show that the magnitude of the impulse is given by

$$J \geq \frac{m}{7}\sqrt{86ga}$$

**Solution:**




M.I. of rod about axis through $C = \frac{1}{3}m \times (2a)^2 + m\left(\frac{3}{2}a\right)^2$ ← From the formula book and using the parallel axes theorem.

$$= \frac{4ma^2}{3} + \frac{9ma^2}{4}$$

$$= \frac{43ma^2}{12}$$

For the impact:

$$J \times \frac{7a}{2} = \frac{43ma^2}{12}\omega$$ ← $\omega$ is the angular speed of the rod.

$$\omega = \frac{42J}{43ma}$$

Energy to top:

$$\frac{1}{2}\left(\frac{43ma^2}{12}\right) \times \left(\frac{42J}{43ma}\right)^2 - \frac{1}{2}\left(\frac{43ma^2}{12}\right)\omega_1^2$$ ← $\omega_1$ is the angular speed of the rod when $B$ is vertically above $A$.

$$= mg \times 3a$$

For complete circles $\omega_1 \ge 0$

$$\therefore \frac{1}{2}\left(\frac{43ma^2}{12}\right) \times \left(\frac{42J}{43ma}\right)^2 - 3mga \ge 0$$

$$\frac{1}{2} \times \frac{43}{12} \times \frac{42^2 J^2}{43^2 m} - 3mga \ge 0$$

$$J^2 \ge 3m^2 ga \times \frac{43 \times 24}{42^2}$$

$$J^2 \ge \frac{86}{49}m^2 ga$$

$$J \ge \frac{m}{7}\sqrt{86\, ga}$$ ← $J > 0$

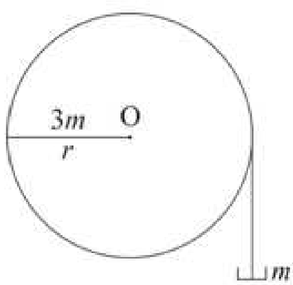
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 11**

**Question:**

A light inextensible string has one end attached to the rim of a pulley wheel of mass $3m$ and radius $r$. The string is wound several times around the wheel. A pan of mass $m$ is attached to the other end of the string and hangs freely below the wheel. The system is held at rest. A particle of mass $5m$ is dropped from rest at a height $4r$ vertically above the pan. The particle adheres to the pan. The wheel is released from rest at the instant the particle hits the pan and begins to rotate about a fixed smooth horizontal axis through the centre of the wheel and perpendicular to the plane of the wheel. Assuming that the pulley wheel can be modelled as a uniform circular disc and the pan as a particle, find an expression for the angular speed of the wheel immediately after the impact.

**Solution:**




For the particle falling freely under gravity:

$s = 4r \quad v^2 = u^2 + 2as$

$a = g \quad v^2 = 2 \times 4rg = 8rg$

$u = 0$

M.I. of the wheel $= \frac{1}{2} \times 3mr^2$ ← From the formula book.

For the impact:

$$5m \times \sqrt{8rg} \times r = \frac{3}{2}mr^2\omega + (5m + m)r\omega \times r$$

← Moment of the initial momentum of the particle = final angular momentum of the wheel + the moment of the final momentum of the pan and particle

$$10\sqrt{2rg} = \frac{3}{2}r\omega + 6r\omega$$

$$10\sqrt{2rg} = \frac{15}{2}r\omega$$

$$\omega = \frac{20}{15}\sqrt{\frac{2g}{r}} = \frac{4}{3}\sqrt{\frac{2g}{r}}$$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise C, Question 12**

**Question:**

A uniform square lamina of mass $m$ and side $2a$ is free to rotate about a fixed smooth horizontal axis which coincides with a side of the lamina. The lamina is hanging in equilibrium when it is hit at its centre of mass by a particle of mass $4m$ moving with speed $v$ in a direction perpendicular to the plane of the lamina. The particle adheres to the lamina.

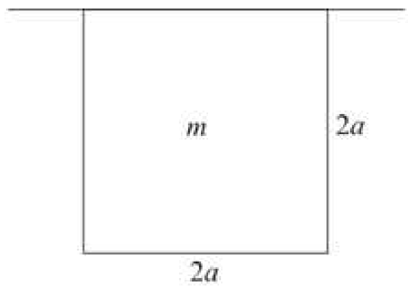
**a** Find the angular speed of the lamina immediately after the impact.

**b** Show that, for the lamina to move in a complete circle,

$$v \geq 22\sqrt{\left(\frac{5ga}{3}\right)}$$

**Solution:**

**a**




M.I. of lamina about axis along a side

$= \frac{4}{3}ma^2$ ← From the formula book.

For the impact:

$4mv \times a = \left(\frac{4}{3}ma^2 + 4ma^2\right)\omega$ ← Angular momentum is conserved.

$4v = \frac{16a\omega}{3}$

$\omega = \frac{3v}{4a}$

The angular speed is $\frac{3v}{4a}$.

**b** Energy:

$\frac{1}{2} \times \frac{16ma^2}{3} \times \left(\frac{3v}{4a}\right)^2 - \frac{1}{2} \times \frac{16ma^2}{3}\omega_1^2$ ← $\omega_1$ is the angular speed when the lamina is vertical and above the axis.

$= 5mg \times 2a$

$\frac{3v^2}{2} - \frac{8a^2}{3}\omega_1^2 = 10ag$

For complete circles, $\omega_1^2 \geq 0$

$\therefore \frac{3v^2}{2} - 10\,ag \geq 0$

$v^2 \geq \frac{2}{3} \times 10ag$

$v^2 \geq \frac{20ag}{3}$

$v \geq 2\sqrt{\frac{5ag}{3}}$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 1**

**Question:**

A simple pendulum is performing small oscillations. Calculate the period of the pendulum when the length is

**a** 2.5 m,
**b** 0.8 m,
**c** 30 cm.

**Solution:**

**a** $T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{2.5}{9.8}}$

$T = 3.173\ldots$

$T = 3.2\text{ s (2 s.f.)}$

**b** $T = 2\pi\sqrt{\frac{0.8}{9.8}} = 1.795\ldots$

$T = 1.8\text{ s (2 s.f.)}$

**c** $T = 2\pi\sqrt{\frac{0.3}{9.8}} = 1.099\ldots$ ← Change cm to m.

$T = 1.1\text{ s (2 s.f.)}$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 2**

**Question:**

A simple pendulum is performing small oscillations. Calculate the length of the pendulum when the period is

**a** $\frac{1}{2}\pi$ s,

**b** $\frac{9}{16}\pi$ s,

**c** 0.8 s.

**Solution:**

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$$

$$l = g\left(\frac{T}{2\pi}\right)^2$$

**a** $l = 9.8\left(\frac{\frac{1}{2}\pi}{2\pi}\right)^2 = \frac{9.8}{16} = 0.6125$

The length is 0.61 m (2 s.f)

**b** $l = 9.8\left(\frac{\frac{9}{16}\pi}{2\pi}\right)^2 = 9.8 \times \left(\frac{9}{32}\right)^2$

$= 0.775\ldots$

The length is 0.78 m (2 s.f)

**c** $l = 9.8\left(\frac{0.8}{2\pi}\right)^2 = 0.1588\ldots$

The length is 0.16 m (2 s.f)

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 3**

**Question:**

A simple pendulum has length $a$ and period $T$. If the length is increased to $2a$, calculate the new period in terms of $T$.

**Solution:**

$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{a}{g}}$

When length is $2a$:

$T' = 2\pi\sqrt{\frac{2a}{g}}$

$= \left(2\pi\sqrt{\frac{a}{g}}\right) \times \sqrt{2}$

$\therefore$ New period is $T\sqrt{2}$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 4**

**Question:**

A seconds pendulum takes one second to perform half an oscillation. Calculate the length of string required for this pendulum.

**Solution:**

Period = 2s

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$2 = 2\pi\sqrt{\frac{l}{g}}$$

$$\left(\frac{1}{\pi}\right)^2 = \frac{l}{g}$$

$$l = \left(\frac{1}{\pi}\right)^2 \times 9.8 = 0.9929\ldots$$

The string must be 0.99 m long (2 s.f.)

1 second for half an oscillation
∴ 2 seconds for a complete oscillation

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 5**

**Question:**

A simple pendulum has length $a$ and period $T$. Calculate, in terms of $a$, the length of a pendulum with period $\frac{1}{2}T$.

**Solution:**

$$T = 2\pi\sqrt{\frac{a}{g}}$$

$$\frac{1}{2}T = 2\pi\sqrt{\frac{l}{g}}$$

$l$ is the length of the pendulum with period $\frac{1}{2}T$.

$$\therefore \frac{1}{2} \times 2\pi\sqrt{\frac{a}{g}} = 2\pi\sqrt{\frac{l}{g}}$$

$$\sqrt{\frac{a}{4g}} = \sqrt{\frac{l}{g}}$$

$$\therefore l = \frac{a}{4}$$

The length is $\frac{a}{4}$.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 6**

**Question:**

One end of a rope is tied to a branch of a tree. A girl is swinging on the other end of the rope. The period of oscillation is 2s. Assuming the girl and the rope can be modelled as a simple pendulum, calculate the length of the rope.
Calculate
**a** the period of small oscillations about the position of stable equilibrium,
**b** the length of the equivalent simple pendulum.

**Solution:**

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$2 = 2\pi\sqrt{\frac{l}{g}}$$

The period is 2s.

$$\left(\frac{2}{2\pi}\right)^2 = \frac{l}{g}$$

$$l = \frac{9.8}{\pi^2} = 0.9929\ldots$$

The length of the rope is 0.99 m (2 s.f.)

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 7**

**Question:**

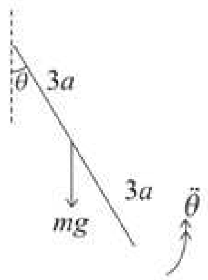
A uniform rod, of mass $m$ and length $6a$, is oscillating about a fixed smooth horizontal axis through one end of the rod.
Calculate

**a** the period of small oscillations about the position of stable equilibrium,

**b** the length of the equivalent simple pendulum.

**Solution:**

**a**




M.I. of rod about axis through one end $= \frac{4}{3}m(3a)^2 = 12ma^2$ ← From the formula book

$mg \times 3a \sin\theta = -12ma^2\ddot{\theta}$ ← Use $L = I\ddot{\theta}$

For small $\theta$, $\sin\theta \approx \theta$

$\therefore 12a^2\ddot{\theta} \approx -3ag\theta$

$\ddot{\theta} \approx -\frac{g}{4a}\theta$ ← The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4a}{g}}$

Period of small oscillations is $4\pi\sqrt{\frac{a}{g}}$

**b** The equivalent simple pendulum has length $4a$. ← Compare $T = 2\pi\sqrt{\frac{4a}{g}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$.

# Solutionbank M5

## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 8**

**Question:**

A uniform rod $AB$ of mass $m$ and length $2a$ with a particle of mass $2m$ attached at $B$, is oscillating about a fixed smooth perpendicular horizontal axis through $A$.

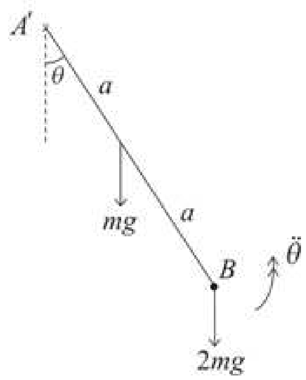
Calculate

**a** the period of small oscillations about the position of stable equilibrium,

**b** the length of the equivalent simple pendulum.

**Solution:**

**a**




M.I. of rod and particle about axis at $A = \frac{4}{3}ma^2 + 2m(2a)^2$

$$= \frac{28}{3}ma^2$$

$$mga\sin\theta + 2mg \times 2a\sin\theta = -\frac{28}{3}ma^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

$$5g\sin\theta = -\frac{28}{3}a\ddot{\theta}$$

For small $\theta$, $\sin\theta \approx \theta$

$$\therefore \ddot{\theta} \approx \frac{-15g}{28a}\theta$$

The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{28a}{15g}}$$

The period of small oscillations is $4\pi\sqrt{\frac{7a}{15g}}$

**b** The equivalent simple pendulum has length $\frac{28}{15}a$

Compare $T = 2\pi\sqrt{\frac{28a}{15g}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

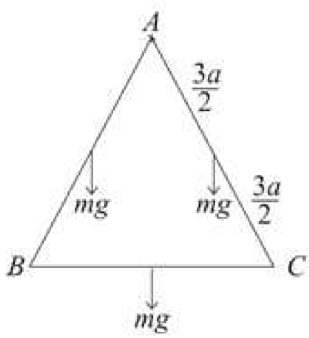
**Exercise D, Question 9**

**Question:**

A triangular framework formed by joining three uniform rods, each of mass $m$ and length $3a$, is oscillating about a fixed smooth horizontal axis through a vertex of the triangle perpendicular to the plane of the triangle.
Calculate
**a** the period of small oscillations about the position of stable equilibrium,
**b** the length of the equivalent simple pendulum.

**Solution:**

**a**



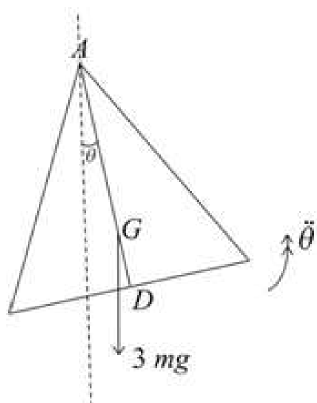

M.I. of framework about axis at $A$

$$= 2\times\frac{4}{3}m\left(\frac{3a}{2}\right)^2+\left[\frac{1}{3}m\left(\frac{3a}{2}\right)^2+m\left\{(3a)^2-\left(\frac{3a}{2}\right)^2\right\}\right]$$

Use the parallel axes theorem to obtain the M.I. of $BC$.

$$= 6ma^2+\frac{3ma^2}{4}+\frac{27ma^2}{4}$$

$$= \frac{27ma^2}{2}$$




Resultant force is $3mg$ at centre of mass of framework.

$AG = \frac{2}{3}AD = \frac{2}{3} \times \frac{3}{2}a\sqrt{3} = a\sqrt{3}$

$\therefore 3mg \times (a\sqrt{3})\sin\theta = \frac{-27}{2}ma^2\ddot{\theta}$

$\ddot{\theta} = -\frac{2g}{9a}(\sqrt{3})\sin\theta$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$\therefore \ddot{\theta} \approx -\frac{2g\sqrt{3}}{9a}\theta$

The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{9a}{2g\sqrt{3}}}$

The period of small oscillations is $6\pi\sqrt{\frac{a}{2g\sqrt{3}}}$.

**b** The equivalent simple pendulum has length $\frac{9a}{2\sqrt{3}}$ or $\frac{3\sqrt{3}}{2}a$

Compare $T = 2\pi\sqrt{\frac{9a}{2g\sqrt{3}}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 10**

**Question:**

A uniform circular disc, of mass $2m$, radius $a$ and centre $O$, with a particle of mass $m$ attached at $A$, where $OA = \frac{1}{2}a$, is oscillating about a fixed smooth horizontal axis through $O$ perpendicular to the disc.

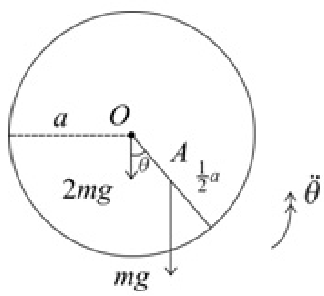
Calculate

**a** the period of small oscillations about the position of stable equilibrium,

**b** the length of the equivalent simple pendulum.

**Solution:**

**a**




M.I. of disc and particle about axis at $O = \frac{1}{2} \times 2ma^2 + m \times \left(\frac{1}{2}a\right)^2$

$$= \frac{5}{4}ma^2$$

$$mg \times \frac{1}{2}a \sin\theta = -\frac{5}{4}ma^2\ddot{\theta}$$ ← Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$$\therefore g\theta \approx -\frac{5}{2}a\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{2g}{5a}\theta$$ ← The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5a}{2g}}$$

The period of small oscillations is $2\pi\sqrt{\frac{5a}{2g}}$.

**b** The equivalent simple pendulum has length $\frac{5a}{2}$.

← Compare $T = 2\pi\sqrt{\frac{5a}{2g}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

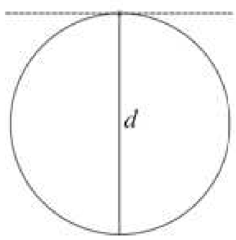
**Exercise D, Question 11**

**Question:**

A uniform circular hoop of mass $m$ and diameter $d$ is oscillating about a fixed smooth horizontal axis coinciding with a tangent to the hoop.
Calculate
**a** the period of small oscillations about the position of stable equilibrium,
**b** the length of the equivalent simple pendulum.

**Solution:**

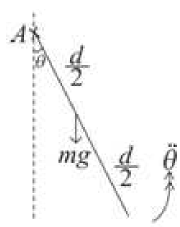
**a**




M.I. of hoop about a diameter $= \frac{1}{2}m\left(\frac{d}{2}\right)^2 = \frac{md^2}{8}$ ← From the formula book and perpendicular axes theorem.

M.I. of hoop about tangential axis $= \frac{md^2}{8} + m\left(\frac{d}{2}\right)^2 = \frac{3md^2}{8}$ ← By the parallel axes theorem.

side view




$mg\left(\frac{d}{2}\right)\sin\theta = -\frac{3md^2}{8}\ddot{\theta}$ ← Use $L = I\ddot{\theta}$

$g\sin\theta = -\frac{3d}{4}\ddot{\theta}$

For small $\theta$ $\sin\theta \approx \ddot{\theta}$ ← The motion is approximately simple harmonic.

$\therefore \ddot{\theta} \approx -\frac{4g}{3d}\theta$

$T = 2\pi\sqrt{\frac{3d}{4g}}$

The period of small oscillations is $\pi\sqrt{\frac{3d}{g}}$

**b** The equivalent simple pendulum has length $\frac{3d}{4}$. ← Compare $T = 2\pi\sqrt{\frac{3d}{4g}}$ with $T = 2\pi\sqrt{\frac{l}{g}}$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 12**

**Question:**

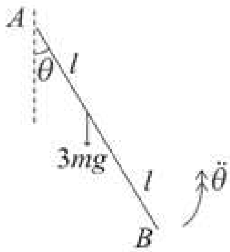
A uniform rod $AB$ of mass $3m$ and length $2l$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through $A$, perpendicular to the plane in which the rod rotates.

**a** Find the period of small oscillations of the rod about its position of equilibrium.

A particle of mass $m$ is now attached to point $B$ of the rod. The period of the oscillations is increased by $x$%.

**b** Find the value of $x$.

**Solution:**

**a** 


M.I. of rod about axis through $A = \frac{4}{3} \times 3ml^2$

$= 4ml^2$

$3mgl \sin\theta = -4ml^2\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$\therefore 3g\theta \approx -4l\ddot{\theta}$

$\ddot{\theta} \approx -\frac{3g}{4l}\theta$

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4l}{3g}} = 4\pi\sqrt{\frac{l}{3g}}$

The period is $4\pi\sqrt{\frac{l}{3g}}$

From the formula book

Use $L = I\ddot{\theta}$

The motion is approximately simple harmonic.

**b** With a particle of mass $m$ at $B$:

$\text{M.I.} = 4ml^2 + m(2l)^2 = 8ml^2$

$3mgl \sin\theta + mg \times 2l \sin\theta = -8ml^2\ddot{\theta}$

$5g\theta \approx -8l\ddot{\theta}$

$\ddot{\theta} \approx -\frac{5g}{8l}\theta$

New period $= 2\pi\sqrt{\frac{8l}{5g}} = 4\pi\sqrt{\frac{2l}{5g}}$

$\therefore \text{\%increase} = \frac{\left(4\pi\sqrt{\frac{2l}{5g}} - 4\pi\sqrt{\frac{l}{3g}}\right)}{4\pi\sqrt{\frac{l}{3g}}} \times 100\%$

$= \frac{\sqrt{\frac{2}{5}} - \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}}} \times 100\%$

$= 9.544\ldots\%$

$\therefore x = 9.54 \text{ (3 s.f.)}$

As in **a** $\sin\theta \approx \theta$

$4\pi$ and $\sqrt{\frac{l}{g}}$ will cancel.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 13**

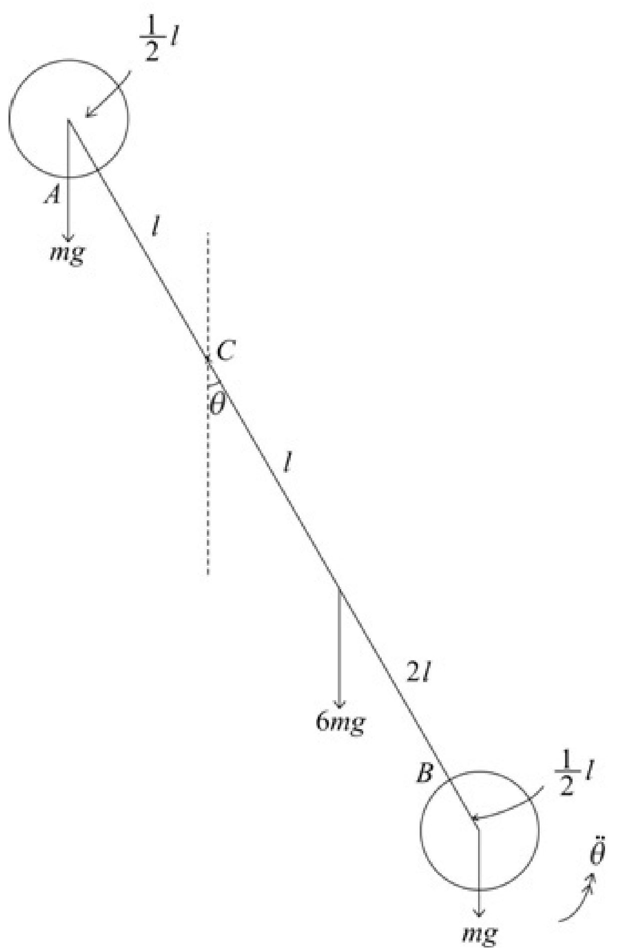
**Question:**

A uniform rod $AB$ of mass $6m$ and length $4l$ has a uniform solid sphere attached to each end. Each sphere has mass $m$ and radius $\frac{1}{2}l$ and the centres of both spheres lie on the same line as the rod. A fixed smooth horizontal axis passes through point $C$ of the rod, where $AC = l$. The rod can rotate in a vertical plane which is perpendicular to this axis.

**a** Show that the moment of inertia of the system about the given axis is $\frac{287ml^2}{10}$.

**b** Find the period of small oscillations of the system about its position of stable equilibrium.

**Solution:**




**a** M.I. of rod about axis thro' $C = \frac{1}{3}(6m)(2l)^2 + 6ml^2$

$= 14ml^2$ ← Use the parallel axes theorem.

M.I. of spheres about axis thro' $C = \frac{2}{5}m\left(\frac{l}{2}\right)^2 + m\left(\frac{3}{2}l\right)^2 + \frac{2}{5}m\left(\frac{l}{2}\right)^2 + m\left(\frac{7l}{2}\right)^2$

$= 2\times\frac{ml^2}{10} + \frac{58ml^2}{4}$

Use the parallel axes theorem again.

$\therefore$ Total M.I. of system

$= 14ml^2 + \frac{ml^2}{5} + \frac{29ml^2}{2}$

$= \frac{(140+2+145)}{10}ml^2$

$= \frac{287}{10}ml^2$

**b** $6mgl\sin\theta + mg\left(\frac{7l}{2}\right)\sin\theta - mg\left(\frac{3l}{2}\right)\sin\theta = -\frac{287ml^2}{10}\ddot{\theta}$

Use $L = I\ddot{\theta}$

$8g\sin\theta = -\frac{287}{10}l\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$\therefore 8g\theta \approx -\frac{287}{10}l\ddot{\theta}$

$\ddot{\theta} \approx -\frac{80g}{287l}\theta$

The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{287l}{80g}}$

The period of small oscillations is $2\pi\sqrt{\frac{287l}{80g}}$ $\left(\text{or } \frac{\pi}{2}\sqrt{\frac{287l}{5g}}\right)$

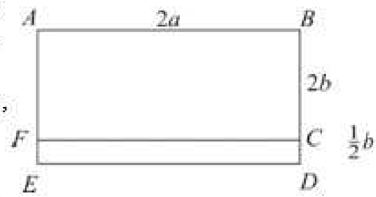
Any equivalent form is acceptable.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 14**

**Question:**

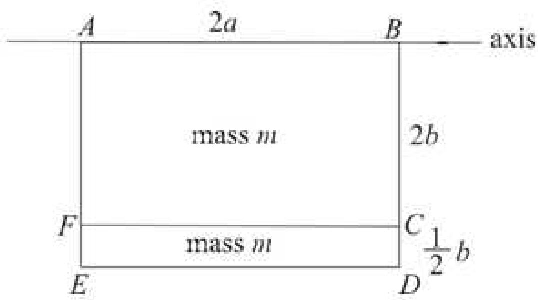
The diagram shows a rectangular sign outside a shop. The sign is composed of two portions, both of which are rectangular. Rectangle $ABCF$ has mass $m$, length $2a$ and width $2b$. Rectangle $FCDE$ has mass $m$, length $2a$ and width $\frac{1}{2}b$. The sign is free to rotate about a fixed smooth horizontal axis which coincides with side $AB$. The wind causes the sign to make small oscillations about its position of stable equilibrium.




Show that the period of these oscillations is given by $2\pi\sqrt{\frac{77b}{39g}}$.

[You may assume that both sections of the sign can be modelled as uniform rectangular laminae.]

**Solution:**




M.I. of sign about axis along $AB$

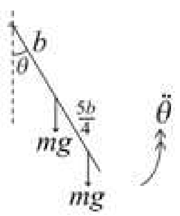
$= \frac{4}{3}mb^2 + \frac{1}{3}m\left(\frac{1}{4}b\right)^2 + m\left(\frac{9}{4}b\right)^2$

$= mb^2\left(\frac{4}{3} + \frac{1}{48} + \frac{81}{16}\right)$

$= \frac{77mb^2}{12}$

Use the parallel axes theorem for $CDEF$. Remember to move from the centre of mass.

Side view




$mgb\sin\theta + mg \times \frac{9b}{4}\sin\theta = -\frac{77}{12}mb^2\ddot{\theta}$

Use $L = I\ddot{\theta}$

For small oscillations, $\sin\theta \approx \theta$

$g\theta + \frac{9g\theta}{4} \approx \frac{-77}{12}b\ddot{\theta}$

$\frac{13}{4}g\theta \approx -\frac{77}{12}b\ddot{\theta}$

$\ddot{\theta} \approx -\frac{39g}{77b}\theta$

The motion is approximately simple harmonic.

Period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{77b}{39g}}$

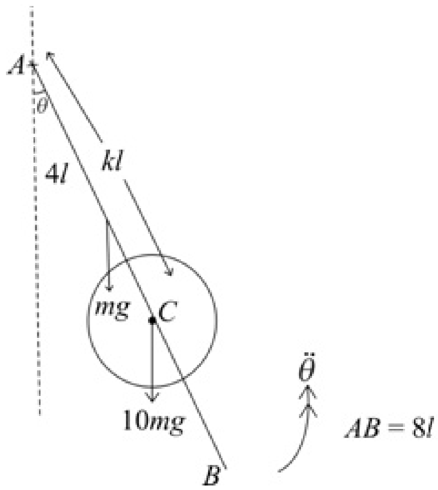
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise D, Question 15**

**Question:**

A thin uniform rod $AB$ of mass $m$ and length $8l$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through end $A$. A uniform circular disc of radius $\frac{1}{2}l$ and mass $10m$ is clamped to the rod with its centre $C$ on the rod and $AC = kl$. The plane of the disc coincides with the plane in which the rod can rotate and the axis is perpendicular to this plane.
Find the length of the equivalent simple pendulum.

**Solution:**




M.I. of rod and disc about axis at $A$

$$= \frac{4}{3}m(4l)^2 + \left[\frac{1}{2}\times 10m\left(\frac{1}{2}l\right)^2 + 10m(kl)^2\right]$$

Use the parallel axes theorem for the disc.

$$= \left(\frac{64}{3} + \frac{5}{4} + 10k^2\right)ml^2$$

$$= \left(\frac{271}{12} + 10k^2\right)ml^2$$

$$mg \times 4l\sin\theta + 10mg \times kl\sin\theta = -\left(\frac{271}{12} + 10k^2\right)ml^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$$4g\theta + 10kg\theta \approx -\left(\frac{271}{12} + 10k^2\right)l\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{(4+10k)g\theta}{\left(\frac{271}{12} + 10k^2\right)l}$$

$$\ddot{\theta} \approx -\frac{24(2+5k)g}{(271+120k^2)l}\theta$$

The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{(271+120k^2)l}{24(2+5k)g}}$$

The equivalent simple pendulum has length $\frac{(271+120k^2)l}{24(2+5k)}$

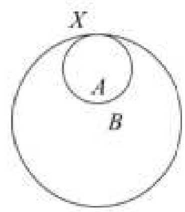
Compare the expression for $T$ with $T = 2\pi\sqrt{\frac{l}{g}}$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

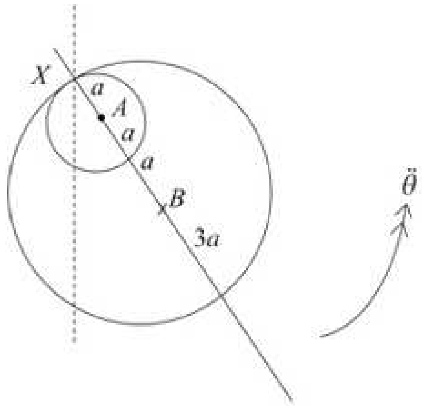
**Exercise D, Question 16**

**Question:**

An ear-ring of mass $8m$ is formed by cutting out a circle of radius $a$ from a thin uniform circular disc of metal, radius $3a$, as shown in the diagram. The centre $B$ of the larger circle, the centre $A$ of the smaller circle and the point $X$ on the circumference of both circles are collinear. The ear-ring is free to rotate about a fixed smooth horizontal axis through $X$ perpendicular to the plane of the ear-ring. Show that the period of small oscillations of the ear-ring about its position of stable equilibrium is $4\pi\sqrt{\frac{15a}{13g}}$.




**Solution:**




| Ratio of areas and masses | Cut-out circle | ear- ring | complete circle |
|---|---|---|---|
| | $\pi a^2$ | $8\pi a^2$ | $9\pi a^2$ |
| | 1 | 8 | 9 |

M.I. of complete disc about axis at $X = \frac{1}{2} \times 9m \times (3a)^2 + 9m \times (3a)^2$

$= \frac{27}{2} m \times (3a)^2 = \frac{243}{2} ma^2$

Use the parallel axes theorem.

M.I. of cut-out circle about axis at $X = \frac{1}{2} ma^2 + ma^2 = \frac{3ma^2}{2}$

$\therefore$ M.I. of ear-ring about axis at $X = \frac{243ma^2}{2} - \frac{3ma^2}{2} = 120ma^2$


You need to find the centre of mass of the ear-ring.

| | | | |
|---|---|---|---|
| Ratio masses | 1 | 8 | 9 |
| Distance from $X$ | $a$ | $\bar{x}$ | $3a$ |

$\therefore 8\bar{x} = 27a - a$

$\bar{x} = \frac{26a}{8}$

$8mg \times \frac{26a}{8} \sin\theta = -120ma^2\ddot{\theta}$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$\therefore 26g\theta \approx -120a\ddot{\theta}$

$\ddot{\theta} \approx \frac{-26g}{120a}\theta$

The motion is approximately simple harmonic.

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{120a}{26g}}$

$= 2\pi\sqrt{\frac{60a}{13g}}$

$= 4\pi\sqrt{\frac{15a}{13g}}$

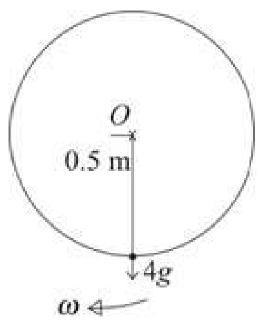
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 1**

**Question:**

A uniform circular disc of mass 20 kg and radius 0.5 m is free to rotate about a fixed smooth horizontal axis through its centre and perpendicular to its plane. A particle of mass 4 kg is attached to a point of the rim of the disc. Initially the disc is at rest in its position of unstable equilibrium. The disc is slightly disturbed. Find the angular speed of the disc at the moment when the particle is vertically below the axis.

**Solution:**




M.I. of disc + particle about axis through $O = \frac{1}{2}\times 20\times(0.5)^2+4\times(0.5)^2$

$= 3.5\,\text{kg m}^2$

Energy:

$\frac{1}{2}\times 3.5\omega^2 = 4g\times 1$

$\omega^2 = \frac{8g}{3.5}$

$\omega = 4.73\ldots$

The angular speed is $4.7\text{ rad s}^{-1}$ (2 s.f.).

The particle starts vertically above $O$ and ends vertically below $O$.
K.E. $=\frac{1}{2}I\omega^2$

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 2**

**Question:**

A uniform rod $AB$ of mass $m$ and length $2a$ is attached to a fixed smooth hinge at $A$. The rod is released from rest with $AB$ horizontal. At time $t$ the angle between the rod and the horizontal is $\theta$.

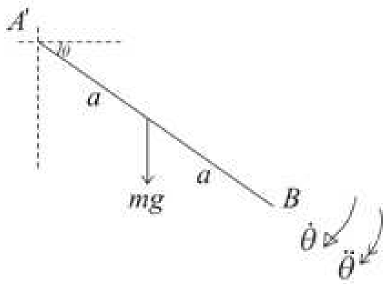
**a** Show that $2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$

**b** Find the magnitude of the component of the force exerted by the rod on the hinge parallel to the rod when $\theta = 45°$.

**c** Find the magnitude of the component of the force exerted by the rod on the hinge perpendicular to the rod when $\theta = 45°$.

**Solution:**

**a**




M.I. of rod about axis through $A = \frac{4}{3}ma^2$

Energy:

$$\frac{1}{2}I\dot{\theta}^2 = mga\sin\theta$$

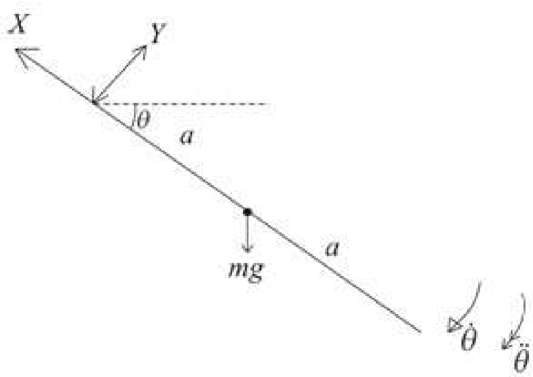
The rod starts from rest with $AB$ horizontal.

$$\frac{2}{3}ma^2\dot{\theta}^2 = mga\sin\theta$$

$$2a\dot{\theta}^2 = 3g\sin\theta$$

$$\text{or } 2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$$

**b**




Equation of motion along the radius:

$$X - mg\sin\theta = ma\dot{\theta}^2$$

For the force on the axis of rotation you need to consider the motion of a particle of mass $m$ at the centre of mass of the rod.

$$X = mg\sin\theta + m \times \frac{3g}{2}\sin\theta$$

When $\theta = 45°$

$$X = mg \times \frac{1}{\sqrt{2}} + \frac{3mg}{2} \times \frac{1}{\sqrt{2}}$$

Use the result from **a**.

$$= \frac{5mg}{2\sqrt{2}}$$

∴ The magnitude of the component of the force exerted by the rod on the hinge parallel to the rod is $\frac{5mg}{2\sqrt{2}}$

**c** Equation of motion perpendicular to the rod:

$mg\cos\theta - Y = ma\ddot{\theta}$

From **a**

$2a\left(\frac{d\theta}{dt}\right)^2 = 3g\sin\theta$

$2\times 2a\frac{d^2\theta}{dt^2} = 3g\cos\theta$ ← $\frac{d}{d\theta}(r\dot{\theta}^2) = 2r\ddot{\theta}$

$\therefore Y = mg\cos\theta - \frac{3mg}{4}\cos\theta$

$\theta = 45°\ Y = \frac{mg}{4}\times\frac{1}{\sqrt{2}} = \frac{mg}{4\sqrt{2}}$

$\therefore$ The magnitude of the component perpendicular to the rod is $\frac{mg}{4\sqrt{2}}$.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 3**

**Question:**

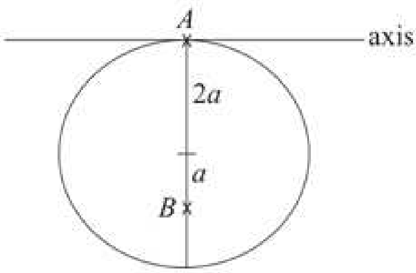
A uniform circular disc of mass $4m$ and radius $2a$ hangs in equilibrium from a point $A$ on its circumference. The disc is free to rotate about a fixed smooth horizontal axis which is tangential to the disc at $A$ and lies in the plane of the disc. A particle $P$ of mass $m$ is moving horizontally towards the disc with speed $V$ in a direction perpendicular to the plane of the disc. The particle strikes the disc at the point $B$ where $AB = 3a$ and $AB$ is perpendicular to the axis. The particle adheres to the disc.

**a** Find the angular speed of the disc immediately after it has been struck by $P$.
The disc first comes to instantaneous rest when the angle between $AB$ and the downward vertical at $A$ is $60°$.

**b** Show that $V = \frac{1}{3}\sqrt{319ga}$ .

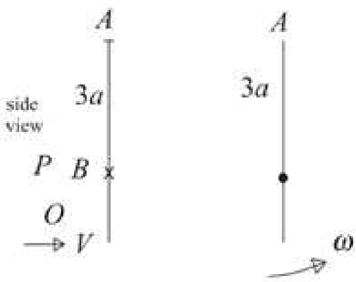
**Solution:**

**a**




M.I. of disc about axis through $A$

$= \frac{1}{4}4m(2a)^2 + 4m(2a)^2 = 20ma^2$ ← Use the parallel axes theorem.



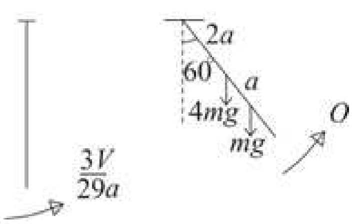

For the impact:

$mV \times 3a = \left(20ma^2 + m \times (3a)^2\right)\omega$ ← Angular momentum is conserved.

$3V = 29a\omega$

$\omega = \frac{3V}{29a}$

**b**




Energy to the highest point:

$$\frac{1}{2}\left(20ma^2 + 9ma^2\right)\left(\frac{3V}{29a}\right)^2 = 4mg \times 2a(1 - \cos 60) + mg \times 3a(1 - \cos 60)$$

$$\frac{1}{2} \times 29ma^2\left(\frac{3V}{29a}\right)^2 = 11mga \times \frac{1}{2}$$

$$\frac{9V^2}{29} = 11ga$$

$$V^2 = \frac{29 \times 11}{9}ga$$

$$V = \frac{1}{3}\sqrt{319\,ga}$$

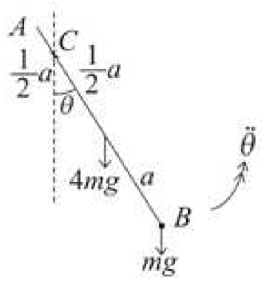
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 4**

**Question:**

A uniform rod $AB$ of mass $4m$ and length $2a$ has a particle of mass $m$ attached at $B$. The rod is free to rotate in a vertical plane about a fixed smooth horizontal axis perpendicular to the rod and passing through point $C$ of the rod where $AC = \frac{1}{2}a$. Find the period of small oscillations of the system about its position of stable equilibrium.

**Solution:**




M.I. of rod and particle about axis through $C$

$$= \left(\frac{1}{3}\times 4ma^2 + 4m\left(\frac{1}{2}a\right)^2\right) + m\left(\frac{3a}{2}\right)^2$$

$$= \frac{4}{3}ma^2 + ma^2 + \frac{9ma^2}{4}$$

$$= \frac{55ma^2}{12}$$

$$4mg \times \frac{1}{2}a\sin\theta + mg\times\frac{3a}{2}\sin\theta = -\frac{55}{12}ma^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

$$\frac{7mga}{2}\sin\theta = -\frac{55}{12}ma^2\ddot{\theta}$$

For small oscillations $\sin\theta \approx \theta$

$$7g\theta \approx -\frac{55}{6}a\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{42g}{55a}\theta \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{55a}{42g}}$$

The motion is approximately simple harmonic.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 5**

**Question:**

A rough uniform rod, of mass $m$ and length $6a$ is held on a rough horizontal table, perpendicular to the edge. A length $2a$ rests on the table and the remainder projects beyond the table.

**a** Find the moment of inertia of the rod about the edge of the table.

The rod is released from rest and rotates about the edge of the table. Assuming that the rod has not started to slip when it has turned through an angle $\theta$,

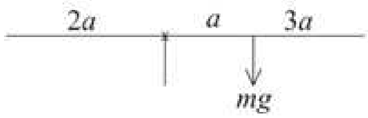
**b** find the angular acceleration of the rod,

**c** find the normal reaction of the table on the rod.

The coefficient of friction between the rod and the edge of the table is $\mu$. The rod starts to slip when it makes an angle $\phi$ with the horizontal.

**d** Find tan $\phi$ in terms of $\mu$.

**Solution:**

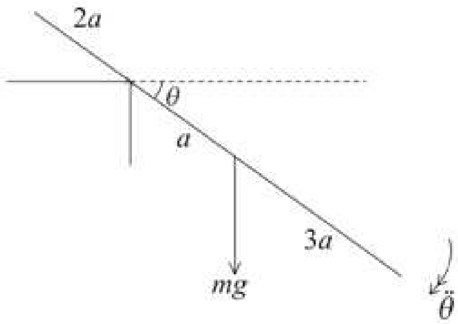



**a** M.I. of rod about edge of table

$= \frac{1}{3}m(3a)^2 + ma^2 = 4ma^2$

From the formula book and using the parallel axes theorem.

**b**

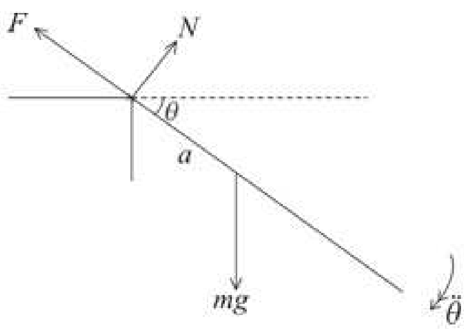



$mga\cos\theta = 4ma^2\ddot{\theta}$

$\ddot{\theta} = \frac{g}{4a}\cos\theta$

Use $L = I\ddot{\theta}$

**c**




Consider the motion of a particle of mass $m$ at the mid-point of the rod.

Equation of motion perpendicular to the rod

$mg\cos\theta - N = ma\ddot{\theta}$

$mg\cos\theta - N = \frac{mg}{4}\cos\theta$

$N = \frac{3}{4}mg\cos\theta$

The normal reaction is $\frac{3}{4}mg\cos\theta$

**d** Equation of motion along the rod:
$F - mg\sin\theta = ma\dot{\theta}^2$

The component of the force parallel to the rod is needed so that you can find $\mu$.

Energy:

$\frac{1}{2}\times 4ma^2\dot{\theta}^2 = mga\sin\theta$

Use conservation of energy to find $\dot{\theta}^2$

$$a\dot{\theta}^2 = \frac{1}{2}g\sin\theta$$

$$\therefore F = \frac{1}{2}mg\sin\theta + mg\sin\theta$$

$$= \frac{3}{2}mg\sin\theta$$

When $\theta = \phi, F = \mu N$

$$\therefore \frac{3}{2}mg\sin\phi = \mu\times\frac{3}{4}mg\cos\phi$$

The rod slips when $\theta = \phi$

$$\tan\phi = \frac{1}{2}\mu$$

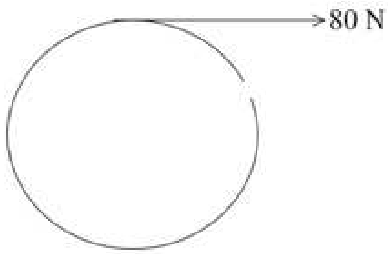
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 6**

**Question:**

A wheel has a rope of length 6 m wound round its axle. The rope is pulled with a constant force of 80 N. When the rope leaves the axle the wheel is rotating at 24 revolutions per minute. Calculate the moment of inertia of the wheel and its axle.

**Solution:**




$$\text{Work done by the force} = 80 \times 6$$
$$= 480 \text{ J}$$

K.E. gained by the wheel

$$= \frac{1}{2} I \omega^2$$

$\omega = 24$ revs. per minute

$$= \frac{24}{60} \times 2\pi = 0.8\pi \text{ rad s}^{-1}$$

$$\therefore \frac{1}{2} I \times (0.8\pi)^2 = 480$$

$$I = \frac{960}{0.64\pi^2} = 151.9\ldots$$

The moment of inertia is $152 \text{ kg m}^2$ (3 s.f.)

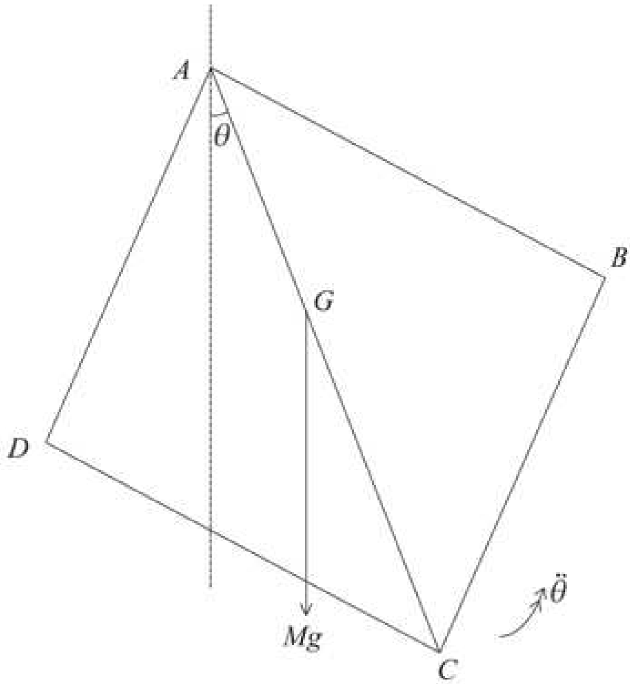
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 7**

**Question:**

A uniform square lamina $ABCD$ of mass $M$ and side $2a$ is free to rotate about a fixed smooth horizontal axis through $A$. The axis is perpendicular to the plane of the lamina. The lamina is hanging at rest with $C$ vertically below $A$. It is then disturbed from rest and performs small oscillations about its position of stable equilibrium. Find the period of these oscillations.

**Solution:**




M.I. of square about axis through $A$

$$= \frac{1}{3}M\left(a^2 + a^2\right) + M\left(a\sqrt{2}\right)^2$$

$$= \frac{2}{3}Ma^2 + 2Ma^2 = \frac{8Ma^2}{3}$$

From the formula book and using the parallel axes theorem $AG = a\sqrt{2}$

$$Mga\sqrt{2}\sin\theta = -\frac{8Ma^2}{3}\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations

$\sin\theta \approx \theta$

$$g\sqrt{2}\theta \approx -\frac{8}{3}a\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{3g\sqrt{2}}{8a}\theta$$

The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{8a}{3g\sqrt{2}}} = 4\pi\sqrt{\frac{a\sqrt{2}}{3g}}$$

The period of small oscillations is $4\pi\sqrt{\frac{a\sqrt{2}}{3g}}$.

# Solutionbank M5

## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 8**

**Question:**

A uniform circular hoop of mass $4m$ and radius $a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis through point $A$ of its circumference. The axis is perpendicular to the plane of the hoop and the hoop is initially hanging in equilibrium. A particle $P$ of mass $m$ is moving horizontally with speed $V$ towards the hoop in the same plane as the hoop. The particle strikes the hoop at one end of its horizontal diameter and adheres to the hoop.

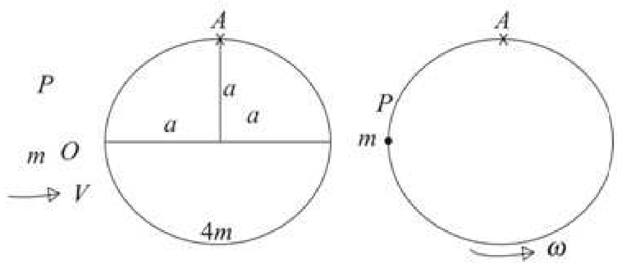
**a** Find the angular speed of the hoop immediately after $P$ strikes it.

The line $AB$ is a diameter of the hoop. The hoop first comes to instantaneous rest when $AB$ is horizontal.

**b** Show that $V^2 = 80ga$

**Solution:**

**a**




M.I. of hoop about axis through $A = 4ma^2 + 4ma^2$
$= 8ma^2$

From the formula book and using the parallel axes theorem.

For the impact:

$mVa = \left(8ma^2 + m\left(a\sqrt{2}\right)^2\right)\omega$

$V = (8a + 2a)\omega$

$\omega = \dfrac{V}{10a}$

Angular momentum is conserved.

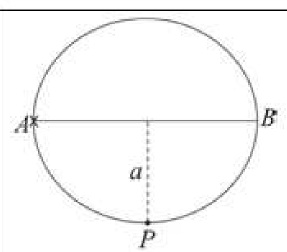
The angular speed is $\dfrac{V}{10a}$.

**b** Energy to highest point:

$\dfrac{1}{2} \times 10ma^2 \left(\dfrac{V}{10a}\right)^2 = 4mga$

$\dfrac{1}{2} \times \dfrac{V^2}{10} = 4ga$

$V^2 = 80\,ga$




The vertical distance from $A$ to $P$ is the same at the start and the end.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 9**

**Question:**

A uniform rod $AB$ of mass $m$ and length $6l$ is free to rotate in a vertical plane perpendicular to a fixed smooth horizontal axis through point $O$ of the rod, where $OA = l$. At time $t = 0$, the rod is at rest in its position of unstable equilibrium and is then slightly disturbed. At time $t$ the rod has turned through an angle $\theta$.

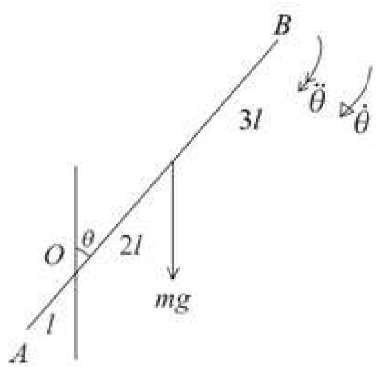
**a** Show that $7l\left(\frac{d\theta}{dt}\right)^2 = 4g(1-\cos\theta)$

**b** Find the magnitude of the angular acceleration of the rod at time $t$.

**c** Calculate the magnitude of the force exerted on the axis when the rod is horizontal.

**Solution:**

**a**




M.I. of rod about axis through $O = \frac{1}{3}m(3l)^2 + m(2l)^2$
$= 7ml^2$

From the formula book and using the parallel axes theorem.

Energy:

$$\frac{1}{2}(7ml^2)\dot{\theta}^2 = mg \times 2l(1-\cos\theta)$$

Increase in K.E. = loss of P.E.

$$7l\dot{\theta}^2 = 4g(1-\cos\theta)$$

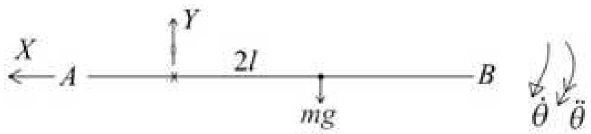
$$\text{or } 7l\left(\frac{d\theta}{dt}\right)^2 = 4g(1-\cos\theta)$$

**b** $mg \times 2l\sin\theta = 7ml^2\ddot{\theta}$

$$\ddot{\theta} = \frac{2g}{7l}\sin\theta$$

Use $L = I\ddot{\theta}$
(or you can differentiate the result from **a** with respect to $t$).

**c**




Consider the motion of a particle of mass $m$ at the centre of mass of the rod.

When the rod is horizontal $\theta = 90^\circ$

$$\therefore 7l\left(\frac{d\theta}{dt}\right)^2 = 4g$$

Use the result from **a**.

$$\text{and } \frac{d^2\theta}{dt^2} = \frac{2g}{7l}$$

Use the result from **b**.

Equation of motion for the particle parallel to $AB$:

$$X = m \times 2l\dot{\theta}^2$$

$$= 2m \times \frac{4g}{7} = \frac{8mg}{7}$$

Equation of motion perpendicular to $AB$:

$$mg - Y = m \times 2l\ddot{\theta}$$

$$Y = mg - 2ml \times \frac{2g}{7l} = \frac{3mg}{7}$$

$$\therefore \text{ Magnitude of the force on the axis} = \frac{mg}{7}\sqrt{(8^2 + 3^2)}$$

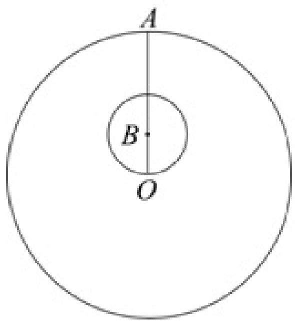
$$= \frac{mg\sqrt{73}}{7}$$

As the magnitude is required, the answer is the same for the force on the rod or the force on the axis.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 10**

**Question:**




A uniform disc of mass $3m$ has centre $O$ and radius $3a$. A disc with centre $B$ and radius $a$ is removed. The line $OB = a$ and, when produced, meets the circumference of the larger disc at $A$ as shown in the diagram. The remaining lamina is free to rotate about a fixed smooth horizontal axis which coincides with the tangent to the disc at $A$.

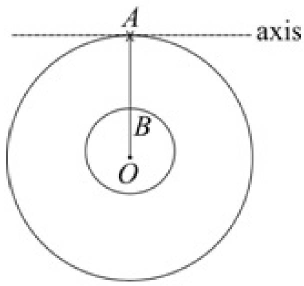
**a** Show that the moment of inertia of the remaining lamina about the given axis is $\frac{97ma^2}{3}$

The lamina is disturbed from rest and makes small oscillations about its position of stable equilibrium.

**b** Find the period of these oscillations.

**Solution:**

**a**




M.I. of complete disc about axis at $A$

$= \frac{1}{4}(3m)(3a)^2 + 3m(3a)^2$

$= \frac{135ma^2}{4}$

From the formula book, and using the parallel axes theorem.

M.I. of cut-out disc about axis at $A$

$= \frac{1}{4}m_1a^2 + m_1(2a)^2$

$= \frac{17}{4}m_1a^2$

$m_1$ is the mass of the smaller disc.

Area of complete disc $= 9\pi a^2$

Area of cut-out part $= \pi a^2$

$\therefore m_1 = \frac{1}{9}$ of mass of complete disc

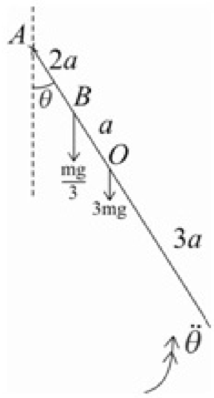
$= \frac{1}{9} \times 3m = \frac{m}{3}$

$\therefore$ M.I. of remaining lamina

$= \frac{135}{4}ma^2 - \frac{17}{4} \times \frac{m}{3}a^2$

$= \frac{97}{3}ma^2$

**b** Side view




Moment of the weight of the lamina about $A$

$$= 3mg \times 3a\sin\theta - \frac{mg}{3} \times 2a\sin\theta$$

$$= \frac{25mga}{3}\sin\theta$$

As you are using moments of forces you do not need to find the centre of mass of the lamina.

$$\therefore \frac{25}{3}mga\sin\theta = -\frac{97}{3}ma^2\ddot{\theta}$$

Use $L = I\ddot{\theta}$

For small oscillations $\sin\theta \approx \theta$

$$25g\theta \approx -97a\ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{25g}{97a}\theta$$

The motion is approximately simple harmonic.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{97a}{25g}} = \frac{2\pi}{5}\sqrt{\frac{97a}{g}}$$

The period of small oscillations is $\frac{2\pi}{5}\sqrt{\frac{97a}{g}}$

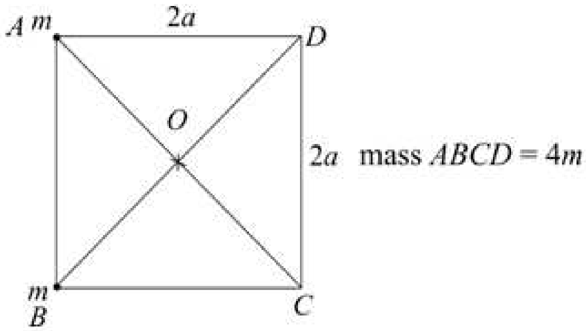
# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 11**

**Question:**

A uniform square lamina $ABCD$ of mass $4m$ and side $2a$ is free to rotate in a vertical plane about a fixed smooth axis through its centre perpendicular to the plane of the lamina. Particles of mass $m$ are attached to vertices $A$ and $B$ of the lamina. The system is released from rest with $AB$ vertical. Find the angular speed of the system when $AB$ is horizontal.

**Solution:**

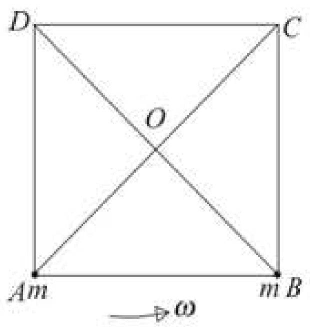



M.I. of lamina and particle about perpendicular axis through $O$

$$= \frac{1}{3} \times 4m(a^2 + a^2) + 2m \times 2a^2$$

$$= \frac{8ma^2}{3} + 4ma^2 = \frac{20ma^2}{3}$$

From formula book (lamina).
$AO^2 = 2a^2$




Energy:

$$\frac{1}{2} \times \frac{20ma^2}{3} \omega^2 = mg \times 2a$$

$$\frac{5}{3} a\omega^2 = g$$

$$\omega^2 = \frac{3g}{5a}$$

$$\omega = \sqrt{\frac{3g}{5a}}$$

Only the particle at $A$ has experienced a change in P.E.

The angular speed is $\sqrt{\frac{3g}{5a}}$.

# Solutionbank M5
## Edexcel AS and A Level Modular Mathematics

**Exercise E, Question 12**

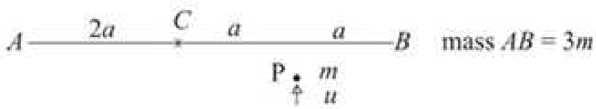
**Question:**

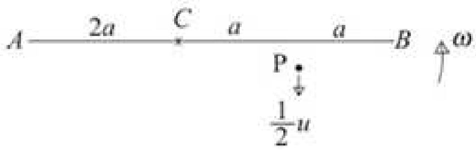
A uniform rod $AB$ of mass $3m$ and length $4a$ lies at rest on a smooth horizontal plane. The rod is free to rotate about a fixed smooth vertical axis through its centre. A particle $P$ of mass $m$ is moving on the table with speed $u$ in a direction perpendicular to the rod. The particle strikes the rod at a distance $a$ from $B$ and rebounds from the rod with its speed half of its speed before the collision.

**a** Find the angular speed of the rod after the collision.

**b** Show that there will not be a second collision between the rod and the particle.

**Solution:**




**a** For the impact:

$$mua = \frac{1}{3}\times 3m(2a)^2\omega - \frac{1}{2}mua$$

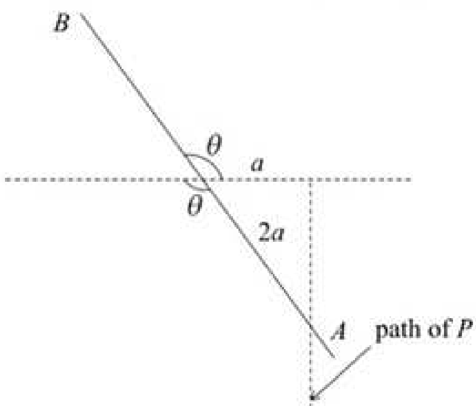
Angular momentum is conserved.

$$4a^2\omega = \frac{3}{2}ua$$

$$\omega = \frac{3u}{8a}$$

The angular speed of the rod is $\frac{3u}{8a}$

**b** Time for the rod to turn through an angle $\theta = \frac{\theta}{\omega} = \frac{8a\theta}{3u}$




Distance travelled by $P$ in this time $= \frac{1}{2}u \times \frac{8a\theta}{3u} = \frac{4a\theta}{3}$

For a second collision, there must be a $\theta, \frac{\pi}{2} < \theta < \pi$ such that

$$\sqrt{\left\{a^2 + \left(\frac{4a\theta}{3}\right)^2\right\}} \le 2a$$

$$1 + \frac{16\theta^2}{9} \le 4$$

$$\frac{16\theta^2}{9} \le 3$$

$$\theta^2 \le \frac{27}{16}$$

$$\therefore \theta \le 1.299$$

but $1.299 < \frac{\pi}{2}$ so there will not be another collision.